Next given a boundary $\partial\Omega$ of a domain Ω, we denote by e_1 the unit tangent vector to both $\partial\Omega$ and Ξ. Let e^1 denote the dual to e_1 in the duality bracket between tangent vectors and co-tangent vectors. Then the area element for $\partial\Omega$ is given by

$$\theta \wedge e^1.$$

Theorem 2: Let M^3 be a simply-connected, complete CR manifold with vanishing torsion tensor. Assume the Webster curvature is non-positive, i.e $W \leq 0$. Then for any domain $\Omega \subseteq M$, we have,

$$\mathrm{vol}(\Omega) \leq c|\partial\Omega|^{4/3}.$$

References

[1] Chanillo,S. and Van Schaftingen, J., Sub-elliptic Bourgain-Brezis Estimates on Groups, Math. Research Letters, **16** (3), (2009), 487-501.

[2] Chanillo, S. and Yang, P. C., Isoperimetric Inequalities and Volume Comparison theorems on CR manifolds, Annali Della Scuola Norm. Sup. Pisa, **8** (2009), 279-307.

On the stability of small crystals under exterior potentials

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The anisotropic isoperimetric inequality arises in connection with a natural generalization of the Euclidean notion of perimeter: given an open, bounded, convex set K of $\mathbb{R}^n$ containing the origin, define a "dual norm" on $\mathbb{R}^n$ by $\|\nu\|_* := \sup\{x \cdot \nu : x \in K\}$. Then, given a (smooth for simplicity) open set $E \subset \mathbb{R}^n$, its anisotropic perimeter is defined as

$$P_K(E) := \int_{\partial E} \|\nu_E(x)\|_* d\mathcal{H}^{n-1}(x).$$

Apart from its intrinsic geometric interest, the anisotropic perimeter P_K arises as a model for surface tension in the study of equilibrium configurations of solid crystals with sufficiently small grains, and constitutes the basic model for surface energies in phase transitions. In the former setting, one is naturally led to minimize $P_K(E)$ under a volume constraint. This is of course equivalent to study the isoperimetric problem (also called Wulff problem [3])

$$\inf\left\{\frac{P_K(E)}{|E|^{(n-1)/n}} : 0 < |E| < \infty\right\}. \tag{1}$$

Introduce the *isoperimetric deficit of E*

$$\delta_K(E) := \frac{P_K(E)}{n|K|^{1/n}|E|^{(n-1)/n}} - 1.$$

This functional measures, in terms of the relative size of the perimeter and of the measure of E, the deviation of E itself from being optimal in (1). The stability problem consists in quantitatively relating this deviation to a more direct notion of distance from the family of optimal sets, which are know to be translations and dilations of K and on which $\delta_K = 0$. In collaboration with F. Maggi and A. Pratelli, we proved the following sharp stability result [2]: if $|E| = |K|$, then

$$C(n)\delta_K(E) \geq \inf_{x\in\mathbb{R}^n}\left(\frac{|E\Delta(x+K)|}{|E|}\right)^2.$$

Here $C(n)$ is an explicit constant, with $C(n) \approx n^7$.

Then, in a joint work with F. Maggi [1] we apply the above improved version of the anisotropic isoperimetric inequality to understand properties of minima arising from the minimization problem

$$E \mapsto P_K(E) + \int_E g, \qquad |E| = m,$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a given potential. The idea is that for small mass m the surface energy P_K dominates, and so one may apply the stability of the isoperimetric problem so say that a minimum F is (quantitatively) close in L^1 to a dilation of K. By combining this starting point with the minimality of F, we show that closedness holds in a quantitative way in some stronger norms: L^∞ for general K, and $C^{k,\alpha}$ if K is smooth and uniformly convex.

A further property that one would like to show is that, for small masses, all minima are actually convex (with no regularity assumptions on g). This last result is shown to be true in two dimension. Moreover, always in two dimensions we proved that if K is polyhedral, then for m small all minima are polyhedral, and their faces are parallels to the ones of K. This (quite surprising) stability property shows that two dimensional crystals are actually very rigid objects.

References

[1] A. Figalli, F. Maggi. Small crystals in external fields. In preparation.
[2] A. Figalli, F. Maggi, A. Pratelli. A mass transportation approach to quantitative isoperimetric inequalities. Submitted.
[3] G. Wulff. Zur Frage der Geschwindigkeit des Wachsturms und der Auflösung der Kristallflächen. *Z. Kristallogr.*, 34, 449-530.

Equilibrium configurations of epitaxially strained crystalline films

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We present some recent results on the equilibrium configurations of a variational model for the epitaxial growth of a thin film on a thick substrate introduced by Bonnetier–Chambolle in [1]. In the model only two dimensional morphologies are considered corresponding to three-dimensional configurations. The reference configuration of the film is

$$\Omega_h = \left\{z = (x,y) \in \mathbb{R}^2 : 0 < x < b,\, 0 < y < h(x)\right\},$$

where $h : [0,b] \to [0,\infty)$ and its graph Γ_h represents the free profile of the film. Denoting by $u : \Omega_h \to \mathbb{R}^2$ the planar displacement of the film with respect to the