The isoperimetric inequality in the Gauss space

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The Gauss measure is a probability measure on $\mathbb{R}^n$ defined by setting for any measurable set $E \subset \mathbb{R}^n$

$$\gamma_n(E) = \frac{1}{(2\pi)^{n/2}} \int_E e^{-\frac{|x|^2}{2}}\, dx\,.$$

If E is a set of locally fnite perimeter, the Gaussian perimeter of E is defined as

$$P_\gamma(E) = \frac{1}{(2\pi)^{n/2}} \int_{\partial^* E} e^{-\frac{|x|^2}{2}}\, d\mathcal{H}^{n-1}(x)\,,$$

where $\partial^* E$ stands for the essential boundary of E in the sense of De Giorgi and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Clearly, both γ_n and P_γ are invariant by rotations around the origin. As in the Euclidean case, also the Gaussian perimeter can be characterized in a variational form. Namely, one has

$$P_\gamma(E) = \sup\Big\{ \int_E \big(\mathrm{div}\varphi(x) - x\cdot\varphi\big) d\gamma_n :\ \varphi \in C^1_0(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_\infty \le 1 \Big\}\,.$$

It is well known that if E is a set such that $\gamma_n(E) = r \in (0,1)$, then

$$P_\gamma(E) \ge P_\gamma(H_{\nu,s})\,, \tag{1}$$

where $\nu \in \mathbb{S}^{n-1}$ and $H_{\nu,s}$ is the half-space $H_{\nu,s} = \{x :\ x\cdot\nu > s\}$ such that

$$r = \gamma_n(H_{\nu,s}) = \frac{1}{\sqrt{2\pi}} \int_s^\infty e^{-t^2/2}\, ds := \Phi(s)\,.$$

Using the function Φ, inequality (1) may be restated as

$$P_\gamma(E) \ge \frac{1}{\sqrt{2\pi}} e^{-[\Phi^{-1}(\gamma_n(E))]^2/2}\,.$$

The first proofs of the Gauss isoperimetric inequality (1) appeared in [6] and [1], followed later by different ones, both of geometric and probabilistic nature (see e.g. the references in [3]). However, only recently it was proved by Carlen and Kerce ([2]) that half-spaces are the only sets for which equality holds in (1). Their proof makes use of probabilistic arguments involving the Ornstein-Uhlenbeck semigroup. We present here a variational proof following the old idea of Steiner to deduce the isoperimetric inequality in the Euclidean case by a symmetrization argument. The analog in the Gauss space of the Steiner symmetrization is the so called Ehrhard symmetrization, first introduced in [4]. More precisely, in [3] the Gaussian isoperimetric inequality (1), together with the characterization of the equality cases, is quickly obtained by proving that the Gaussian perimeter strictly decreases under the Ehrhard symmetrization of a set E in a given direction $\nu \in \mathbb{S}^{n-1}$, unless the one dimensional sections of E parallel to ν are half-lines or lines.

By using Ehrhard symmetrization in [3] we prove also a quantitative version of inequality (1). In fact we show that the stronger inequality holds

$$P_\gamma(E) \geq P_\gamma(H_{\nu,s}) + \frac{\lambda^2(E)}{C^2(n,r)}\,, \tag{2}$$

where $\lambda(E)$ is the asymmetry index of the set E,

$$\lambda(E) = \min_{\nu\in\mathbb{S}^{n-1}} \left\{\gamma_n\big(E\triangle H_{\nu,s}\big) :\ \gamma(H_{\nu,s}) = \gamma_n(E) = r\right\}.$$

The quantitative inequality (2) can be also rewritten as

$$\lambda(E) \leq C(n,r)\sqrt{\delta(E)}\,,$$

where $\delta(E) = P_\gamma(E) - P_\gamma(H_{\nu,s})$ is the isoperimetric deficit of E.
Inequality (2) extends to the Gaussian context the quantitative (Euclidean) isoperimetric inequality proved in [5]

$$\Lambda^2(E) \leq C(n)\sqrt{\Delta(E)}\,,$$

where $\Lambda(E)$ is the Fraenkel asymmetry of E

$$\Lambda(E) = \min_{x\in\mathbb{R}^n}\left\{\frac{|E\triangle B_r(x)|}{|E|} :\ |E| = |B_r|\right\}$$

and $D(E)$ is the isoperimetric deficit

$$D(E) = \frac{P(E) - P(B_r)}{P(B_r)}\,,$$

$P(E)$ and $P(B_r)$ being the Euclidean perimeter of E and of a ball of radius r, respectively.

References

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