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Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law

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(joint work with G. De Philippis)

The classical description of capillarity phenomena involves the study of Gauss free energy for a liquid inside a container, which takes the form

$$\mathcal{H}^{n-1}(A\cap\partial E)+\int_{\partial A\cap\partial E}\sigma(x)\,d\mathcal{H}^{n-1}(x)+\int_E g(x)\,dx-l\,|E|\,.$$

Here A is an open set in $\mathbb{R}^n$ ($n\ge 2$), the container of the fluid; $E\subset A$ is the region occupied by the fluid, with volume $|E|$; $\mathcal{H}^{n-1}(A\cap\partial E)$ is the total surface tension energy of the interior interface $A\cap\partial E$; the surface tension between the liquid and the boundary walls of the container is obtained by integrating over the wetted surface $\partial A\cap\partial E$ the coefficient $\sigma(x)$; finally, $g(x)$ is the potential energy density (typically, when $n=3$ one considers $g(x)=\rho\,g_0\,x_3$ where ρ is the constant density of the fluid and g_0 the gravity of Earth), and l is a Lagrange multiplier. If $M=A\cap\partial E$ is smooth enough, then the equilibrium conditions are

$$H_E+g=l\qquad\text{on } A\cap\partial E\,,\tag{1}$$

$$\nu_E\cdot\nu_A=\sigma\qquad\text{on } \partial A\cap\partial E\,,\tag{2}$$

where ν_E is the outer unit normal to E and H_E is the mean curvature of $A\cap\partial E$. These conditions, first described by Young in [12], have then been expressed in analytic form by Laplace in 1805; see [5]. The second condition, commonly known as *Young's law*, enforces $|\sigma|\le 1$ and is independent from the potential energy g.

Volume constrained minimizers of the Gauss free energy are found in the class of sets of finite perimeter. One is thus lead to discuss a regularity problem in order to validate (1) and (2). Interior regularity has been addressed in the classical theory developed by De Giorgi, Federer, Almgren, and others in the Sixties: if ∂^*E denotes the reduced boundary of the minimizer E, and we set $M=\text{closure}(A\cap$

$\partial^* E$), then there exists a closed set $\Sigma \subset M$ such that $M \setminus \Sigma$ is "as smooth as g allows it to be" and Σ has Hausdorff dimension at most $n-8$. The situation concerning boundary regularity is less conclusive. Taylor [11] proved in dimension $n = 3$ everywhere regularity of M at ∂A (in the more general context of $(\mathbf{M}, \xi, \delta)$-minimal sets). Cafferelli and Friedman [2] addressed the sessile droplet problem ($A = \{x_n > 0\}$ and $g(x) = g(x_n)$) in the case when $-1 < \sigma(x) < 0$ for every $x \in \{x_n = 0\}$ and $2 \leq n \leq 7$ by mixing symmetrization arguments, barrier techniques, interior regularity for perimeter minimizers, and the regularity theory of free boundary problems associated to quasilinear uniformly elliptic equations. Grüter [7, 8, 9] and Grüter and Jost [6] addressed the case when $\sigma \equiv 0$ by exploiting reflection techniques and interior regularity.

Motivated by applications to relative isoperimetric problems in Riemannian and Finsler geometry, one would also like to understand the regularity of minimizers of anisotropic surface energies of the form

$$I(E) = \int_{A \cap \partial E} \Phi(\nu_E)\, d\mathcal{H}^{n-1} + \int_{\partial A \cap \partial E} \sigma\, d\mathcal{H}^{n-1}$$

where $\Phi : A \times \mathbb{R}^n \to [0, \infty)$ is such that $\Phi(x, \cdot)$ positively one-homogeneous and convex on $\mathbb{R}^n$ for every $x \in A$. The typical assumption to obtain regularity here is that $\Phi(x, \nu)$ is l-elliptic in ν: roughly speaking, one asks that for some $l \in (0, 1]$, and for every $x \in A$ and $\nu \in S^{n-1}$,

$$l \leq \Phi(x, \nu) \leq \frac{1}{l}\,, \qquad \nabla^2 \Phi(x, \nu) \geq l\mathrm{Id} \qquad \text{on } \nu^\perp\,.$$

Under this assumption, interior regularity is known since the works of Almgren [1], and Schoen, Simon and Almgren [10]. In [3, 4] we address boundary regularity.

Theorem 1. *Let $\partial A \in C^{1,1}$, Φ be l-elliptic in ν and uniformly Lipschitz in x, let $\sigma \in \mathrm{Lip}(\mathbb{R}^n)$ be such that $-\Phi(x, -\nu_A) < \sigma(x) < \Phi(x, \nu_A)$ for $x \in \partial A$, and let $E \subset A$ be such that*

$$I(E) \leq I(F) + \Lambda\, |E \Delta F|\,,$$

whenever $F \subset A$, $E \Delta F \subset\subset B_{x,r}$ where $x \in A$ and $r < \delta$. Then E is equivalent to an open set, $\partial E \cap \partial A$ is a set of finite perimeter in ∂A, and there exists a closed set $\Sigma \subset \mathrm{closure}(A \cap \partial E) =: M$ such that $M \setminus \Sigma$ is a $C^{1,1/2}$-manifold with boundary, $\mathcal{H}^{n-3}(\Sigma) = 0$, and the (anisotropic) Young's law

$$\nabla \Phi(x, \nu_E(x)) \cdot \nu_A(x) = \sigma(x)\,,$$

holds for every $x \in (M \setminus \Sigma) \cap \partial A$.

As said, the fact that $A \cap (M \setminus \Sigma)$ is a $C^{1,1/2}$-manifold for a closed set $\Sigma \subset M$ with $\mathcal{H}^{n-3}(A \cap \Sigma) = 0$ is proved in [1, 10]: our contribution here is addressing the situation at boundary points, namely, on $M \cap \partial A$. As explained above, this last problem was still partially open in the isotropic case, and, to the best of our knowledge, completely open in the genuinely anisotropic case. In [3] we have proved Theorem 1 in a weaker form, where one only concludes that $\mathcal{H}^{n-2}(\partial A \cap \Sigma) = 0$; starting from this dimensional estimate, in [4] we have further developed our analysis to conclude that $\mathcal{H}^{n-3}(\partial A \cap \Sigma) = 0$, thus matching the best known interior regularity results.

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The structure of minimum matchings

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(joint work with Roger Züst)

1. Calibrations without orientation

We recall here the setting of the theory of calibrations (see [5], [4]). The following is a simple proof that the shortest oriented curve connecting two points $a, b \in \mathbb{R}^n$ is the oriented segment $[a, b]$. Let α be the constant coefficient differential 1-form dual to the unit vector τ orienting $[a, b]$. Then for any other Lipschitz curve γ from a to b we have

$$\text{lenght}([a,b]) = \int_{[a,b]} \alpha = \int_\gamma \alpha \leq \text{lenght}(\gamma) \ , \tag{1}$$