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Isoperimetry with upper mean curvature bounds and sharp stability estimates

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(joint work with Francesco Maggi)

Motivated by capillarity-type problems, in our recent work of [7], we consider the structure of hypersurfaces with almost constant mean curvature (almost CMC). For a bounded, open subset $\Omega \subset \mathbb{R}^{n+1}$ with a smooth boundary, we define the *CMC deficit* $\delta_{\rm cmc}(\Omega)$ by

$$\delta_{\rm cmc}(\Omega) = \left\| \frac{H_\Omega}{H_0} - 1 \right\|_{L^\infty(\partial\Omega)} \quad \text{where} \quad H_0 = \frac{n\,P(\Omega)}{(n+1)\,|\Omega|}$$

and where H_Ω denotes the mean curvature of $\partial\Omega$ computed with respect to the outward unit normal to Ω and $P(\,\cdot\,)$ denotes perimeter. Note that if $\partial\Omega$ is CMC, then $H_\Omega = H_0$. We say that $\partial\Omega$ is almost CMC if $\delta_{\rm cmc}(\Omega)$ is small.

Previous work by Ciraola and Maggi [3] showed that if $\partial\Omega$ is almost CMC, $H_0 = n$, and $P(\Omega) \leq (L+\tau)\,P(B_1)$ for an integer $L \geq 1$ and $\tau \in (0,1)$, then $\partial\Omega$ can be represented as a $C^{1,\alpha}$ graph over a union of at most L tangent unit balls away from spherical caps where the tangent balls touch and with estimates. However, the estimates of [3] were not optimal. Ciraolo and Vezzoni [5] showed that if $\partial\Omega$ is almost CMC, $|\Omega| = |B_1|$, and Ω satisfies an interior/exterior ball condition of radius $\rho > 0$ at each point of $\partial\Omega$, then $\mathrm{hd}(\partial\Omega, \partial B_1(x_0)) \leq C(n, P(\Omega), \rho)\,\delta_{\rm cmc}(\Omega)$ for some $x_0 \in \mathbb{R}^{n+1}$. This estimate is optimal. However, the interior/exterior ball condition is too restrictive for the study of critical points of capillarity-type energies since the uniform ball condition prevents bubbling phenomena; for instance, consider the surface obtained by truncating and then smoothly completing an unduloid with very thin necks. As a step towards obtaining sharp estimates for almost CMC hypersurfaces close to a union of tangent unit balls, we will prove sharp estimates for almost CMC hypersurfaces close to a single unit ball without the interior/exterior ball condition.

Our approach involves an isoperimetric principle due to Almgren [1], which in codimension one can be stated as follows:

Almgren's isoperimetric principle. *If $\Omega \subset \mathbb{R}^{n+1}$ is a bounded, open subset with a smooth boundary such that $H_\Omega \leq n$ at each point of $\partial\Omega$, then $P(\Omega) \geq P(B_1)$ with equality if and only if Ω is a unit ball.*

This isoperimetric principle arises from Almgren's proof of the sharp isoperimetric inequality in codimension > 1 and can be stated more generally for weak notions of submanifolds, namely rectifiable varifolds, in arbitrary codimension. In codimension one, mean curvature can be represented by a scalar quantity and we assume a one-sided bound $H_\Omega \leq n$ with no lower bound on H_Ω. Almgren's proof of the isoperimetric principle yields the quantitative information that $\delta(\Omega) = P(\Omega) - P(B_1)$ is given by

$$\delta(\Omega) = \int_{\partial A \cap \partial\Omega} \left(\left(\frac{H_\Omega}{n} \right)^n - K_\Omega \right) + \int_{\partial A \cap \partial\Omega} \left(1 - \left(\frac{H_\Omega}{n} \right)^n \right) + \mathcal{H}^n(\partial\Omega \cap \partial A),$$

where A is the convex hull of Ω and each integrand is nonnegative. This naturally leads to the question of whether one can use this quantitative information to address stability for Almgren's isoperimetric principle: *If $\Omega \subset \mathbb{R}^{n+1}$ is a bounded, open subset with a smooth boundary such that $H_\Omega \leq n$ on $\partial\Omega$ and $\delta(\Omega) = P(\Omega) - P(B_1)$ is small, must Ω be close to a unit ball? Can we obtain sharp estimates on $\mathrm{hd}(\partial\Omega, \partial B_1)$ and $|\Omega \,\Delta\, B_1|$ in terms of $\delta(\Omega)$?*

A key obstruction to stability for Almgren's isoperimetric principle is that, since we do not assume any lower bound on H_Ω, Ω could have tiny holes, e.g. $\Omega = B_1 \setminus B_\varepsilon$, or thin tentacles protruding into Ω. As a result, we could have $\mathrm{hd}(\partial\Omega, \partial B_1) \approx 1$ despite $\delta(\Omega)$ being small. In [7, Theorem 1.2], we remove the holes and tentacles by using Almgren's isoperimetric principle to show that the total perimeter and volume of the holes is $\leq C(n)\,\delta(\Omega)$ and by replacing Ω with a solution E to an obstacle problem, minimize $P(E) + n\,|E|$ amongst E with $\Omega \subseteq E$, as the minimizer E satisfies $-n \leq H_E \leq n$ a.e. on ∂E and has total perimeter and volume outside of Ω that is $\leq C(n)\,\delta(\Omega)$.

Now assuming that $-n \leq H_\Omega \leq n$ on $\partial\Omega$, one can argue using Allard regularity that if $\delta(\Omega)$ is sufficiently small then, up to translating, $\Omega = \{(1 + u(x))\,x : x \in \mathbb{S}^n\}$ for some $u \in C^1(\mathbb{S}^n)$ with $\|u\|_{C^1} \leq \varepsilon(n)$ small. Having shown this, we obtain the following sharp stability estimates for Almgren's isoperimetric inequality.

Theorem 1 (Theorem 1.1 of [7])**.** *If $\Omega \subseteq \mathbb{R}^{n+1}$ is a bounded, open set with smooth boundary such that $H_{\partial\Omega} \leq n$ on $\partial\Omega$ and $\delta(\Omega) \leq \delta_0(n)$, then there exists $x_0 \in \mathbb{R}^{n+1}$ such that*

$$|\Omega \,\Delta\, B_1(x_0)| + \sup\{s > 0 : \Omega \subseteq B_{1+s}(x_0)\} \leq C(n)\,\delta(\Omega).$$

Observe that, by contrast, the isoperimetric inequality has a different stability estimate, $|\Omega \,\Delta\, B_1(x_0)|^2 \leq C(n)\,\delta(\Omega)$.

Theorem 2 (Theorem 1.5 of [7])**.** *If $\Omega \subseteq \mathbb{R}^{n+1}$ is a bounded, open set with smooth boundary such that $H_{\partial\Omega} \leq n$ on $\partial\Omega$, $\partial\Omega = \{(1 + u(x))\,x : x \in \mathbb{S}^n\}$ for* *some* $u \in C^1(\mathbb{S}^n)$ *with* $\|u\|_{C^1} \leq \varepsilon(n)$*, and* $\int_{\partial\Omega} x\, d\mathcal{H}^n(x) = 0$*, then*

$$\mathrm{hd}(\partial\Omega, \mathbb{S}^n) \leq \begin{cases} C(1)\,\delta(\Omega) & \text{if} n = 1 \\ C(2)\,\delta(\Omega)\,\log(C(2)/\delta(\Omega)) & \text{if} n = 2 \\ C(n)\,\delta(\Omega)^{\frac{1}{n-1}} & \text{if} n \geq 3. \end{cases} .$$

Moreover, there exists an example showing that this estimates is sharp.

The proofs of Theorems 1 and 2 use a series expansion argument based on [6], which when combined with elliptic estimates and an interpolation inequality of Fuglede [6, Lemma 1.4] yields the estimates in Theorems 1 and 2. The series expansion argument additionally shows that the average of u dominates the other Fourier coefficients of u so that

$$0 < c(n)\,\delta(\Omega) \leq \int_{\mathbb{S}^n} u \leq C(n)\,\delta(\Omega).$$

Since taking u to be constant corresponds to scaling the unit sphere, we interpret this as meaning that we must scale the unit sphere outward while deforming it into $\partial\Omega$ in order to preserve $H_\Omega \leq n$. Note that, by contrast, for the isoperimetric inequality one typically fixes the volume of Ω and consequently the average of u is negligible, i.e. $\int_{\mathbb{S}^n} u = O(\|u\|^2_{W^{1,2}})$.

We construct the example showing that the Hausdorff distance estimates in Theorem 2 are sharp as follows. Rescale the unit sphere by $1 + t$ for $t > 0$ small. Push the rescaled sphere in at the north pole to form a tentacle as a surface of revolution. Up to radius r_1 from the axis of symmetry for the tentacle, the profile the tentacle will be a solution to an ordinary differential equation and the tentacle roughly behaves like the graph of a fundamental solution for the Laplacian. We cut this portion of the tentacle off at a radius r_1 where its gradient relative to the unit sphere equals a small constant $\mu > 0$ and then cap off the tentacle with a spherical cap.

Our approach also yields the following sharp estimates for almost CMC hypersurfaces close to a single tangent ball, see [7]. Let $\partial\Omega = \{(1 + u(x))\, x : x \in \mathbb{S}^n\}$ for some $u \in C^1(\mathbb{S}^n)$ with $\|u\|_{C^1}$ small and $\int_{\partial\Omega} x\, d\mathcal{H}^n(x) = 0$. If $\|H_\Omega - n\|_{L^2(\partial\Omega)}$ is sufficiently small, then $\|u\|_{W^{1,2}} \leq C(n)\,\|H_\Omega - n\|_{L^2(\partial\Omega)}$. This result is of particular interest since it may have applications to convergence to equilibrium in geometric flows, see for instance [4] for this kind of application of stability theorems to Yamabe-type fast diffusion equations). If additionally

$$\delta^{(p)}_{\mathrm{cmc}}(\Omega) = H_0^{-1} \max\left\{\|(H_0 - H_\Omega)^+\|_{L^\infty(\partial\Omega)}, \|(H_\Omega - H_0)^+\|_{L^p(\partial\Omega)}\right\}$$

is sufficiently small for $p \geq 2$ when $n = 2$ and $p > n/2$ when $n \geq 3$, then $\mathrm{hd}(\partial\Omega, \mathbb{S}^n) \leq C(n,p)\,\delta^{(p)}_{\mathrm{cmc}}(\Omega)$.

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Sharp local smoothing estimates for the Ricci flow on surfaces.

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(joint work with Hao Yin)

Consider the logarithmic fast diffusion equation

$$\partial_t u = \Delta \log u \tag{1}$$

where u is a positive function on a two-dimensional domain. This equation has an extensive literature when posed on the plane, and many beautiful results. The logarithm makes the equation nonlinear, but in a very special way. The geometric reason why this particular choice of equation is so natural comes from the fact that it describes locally the Ricci flow on surfaces. In this two-dimensional situation, the Ricci flow is a one parameter family of Riemannian metrics $g(t)$ that is governed by the nonlinear evolution equation

$$\frac{\partial g}{\partial t} = -2Kg$$

where K is the Gauss curvature of g. In local isothermal coordinates x, y, we can write $g = u.(dx^2 + dy^2)$, and then u can be seen to satisfy the logarithmic fast diffusion equation.

We are interested in posing the Ricci flow with very rough data. For smooth initial data, there is an extensive literature, with the ultimate result that we have existence of a unique instantaneously complete solution for completely arbitrary (smooth) initial data, with that solution existing for a definite period of time, normally for all time (see [1] and [3]). To make a Ricci flow with very general (e.g. locally L^1) initial data, there is an obvious strategy that is to approximate the initial data by smooth initial data, each of which is then Ricci flowed with the existing theory, and then to try to pass to a limit of the flows. To have any hope of getting reasonable convergence of these smooth flows to the desired flow, we need uniform C^k estimates at later times, and by standard parabolic theory, we can obtain these if we are able to derive uniform L^∞ estimates on $\log u$, i.e uniform estimates on u from above and below by positive numbers depending on the time at which we demand an estimate, but independent of how good an approximation of the initial data we have taken. Lower bounds on u are relatively easy to obtain using the global theory in [1], but upper bounds are more tricky. Indeed, for L^1