from the kernel of $\mathcal{I}_4$ remains uniformly bounded away from 0. We note that this lack of coercivity is not prevented by the fact that the kernel is finite dimensional.

Critical and almost-critical points in isoperimetric problems

Francesco Maggi

(joint work with M. Delgadino, C. Mihaila, R. Neumayer)

De Giorgi's isoperimetric inequality [1] states that if Ω is a measurable set with finite volume $|\Omega| < \infty$, then the distributional perimeter $P(\Omega)$ of Ω satisfies

$$P(\Omega) \geq (n+1)\,|B_1|^{1/(n+1)}\,|\Omega|^{n/(n+1)} \qquad B_1 = \{x : |x| < 1\}\,, \tag{1}$$

with equality if and only if Ω is a equivalent to a ball. In other words, *among sets of finite perimeter (SFP) with fixed volume, balls are the unique minimizers of perimeter.* Looking more generally at critical points, rather than at minimizers, for a variation $f_t(x) = x + t\,X(x) + O(t^2)$ with $X \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ we have that

$$\frac{d}{dt}\bigg|_{t=0} |f_t(\Omega)| = \int_\Omega \mathrm{div} X = \int_{\partial^*\Omega} X\cdot\nu_\Omega\,, \qquad \frac{d}{dt}\bigg|_{t=0} P\big(f_t(\Omega)\big) = \int_{\partial^*\Omega} \mathrm{div}^{\partial^*\Omega} X\,.$$

Thus a critical point of perimeter must satisfy, for a constant λ,

$$\int_{\partial^*\Omega} \mathrm{div}^{\partial^*\Omega} X = \lambda \int_{\partial^*\Omega} X\cdot\nu_\Omega\,, \qquad \forall X \in C^1_c\,. \tag{2}$$

Here $\partial^*\Omega$ is the reduced boundary of Ω, ν_Ω the outer unit normal to Ω in the measure theoretic sense, and, necessarily, $\lambda = H^0_\Omega$, where

$$H^0_\Omega = \frac{n\,P(\Omega)}{(n+1)\,|\Omega|}\,.$$

When $\partial\Omega \in C^2$, then (2) is equivalent to $H_\Omega \equiv H^0_\Omega$ along $\partial\Omega$, where H_Ω denotes the mean curvature of Ω (w.r.t. ν_Ω); moreover, in this case, $|\Omega| < \infty$ implies that Ω is bounded (by area monotonicity), and so the moving planes method of Alexandrov [2] can be applied to conclude that *among C^2-sets with fixed volume, balls are the unique critical points of perimeter.* The gap in the characterization of critical points between C^2-sets and finite perimeter sets is addressed in a joint paper with Delgadino [3], where we prove the following theorem.

Theorem [Alexandrov's theorem revisited] *Among sets of finite perimeter with fixed volume, finite unions of balls are the unique critical points of perimeter.*

Wente's torus is a non-shperical example of a 2-dimensional stationary unit density integer rectifiable varifold in $\mathbb{R}^3$ with constant mean curvature (CMC). As an immediate corollary,

Compactness I: *If $\{\Omega_j\}_j$ is a sequence of sets of finite perimeter and finite volume with $\Omega_j \to \Omega$ in L^1, such that there exists a constant λ with*

$$\lim_{j\to\infty} P(\Omega_j) = P(\Omega) \qquad \lim_{j\to\infty} \int_{\partial^*\Omega_j} \Big\{\mathrm{div}^{\partial^*\Omega_j} X - \lambda\, X\cdot\nu_{\Omega_j}\Big\} = 0\,, \tag{3}$$

whenever $X \in C^1_c$, then Ω is a finite union of balls.

This compactness statement is interesting in view of the many variational problems where almost-CMC boundaries arise. Examples include capillarity type problems, CMC-foliations in general relativity, and long-time behavior of weak solutions to mean curvature flows (MCF). Weak solutions to the volume preserving MCF are generally constructed as families of SFP $\{\Omega(t)\}_{t\geq 0}$ with distributional mean curvature $H_t \in L^2(\mathcal{H}^n \llcorner \partial^*\Omega(t))$, and satisfy the dissipation inequality

$$\int_0^\infty dt \int_{\partial^*\Omega(t)} (H_t - \langle H_t\rangle)^2\, d\mathcal{H}^n \leq P(\Omega(0)) < \infty\,. \tag{4}$$

Now assume, for a sequence of times $t_j \to \infty$, that: (i) the averages $\langle H_{t_j}\rangle$ are bounded; (ii) there exists Ω such that $\Omega(t_j) \to \Omega$ in L^1 and $P(\Omega(t_j)) \to P(\Omega)$; and (iii) exploiting (4), and up to extracting subsequences, that

$$\lim_{j\to\infty} \int_{\partial^*\Omega(t_j)} (H_{t_j} - \langle H_{t_j}\rangle)^2\, d\mathcal{H}^n = 0\,.$$

Then Ω is necessarily a finite union of balls, thanks to Compactness I. The assumption $P(\Omega_j) \to P(\Omega)$ in Compactness I can be dropped if working with smooth sets with H_{Ω_j} converging to a constant in L^2, and satisfying a uniform mean convexity bound. More precisely, in a joint paper with Delgadino, Mihaila and Neumayer [4] we proved:

Compactness II: *If $\{\Omega_j\}_j$ is a sequence of open sets with smooth boundary and finite volume, normalized by scaling so to have $H^0_{\Omega_j} = n = H_{B_1}$ and such that for a constant $\kappa > 0$*

$$H_{\Omega_j} \geq \kappa \text{ on } \partial\Omega_j$$

then

$$\Omega_j \to \Omega \text{ in } L^1 \qquad \text{and} \qquad \lim_{j\to\infty} \int_{\partial\Omega_j} |H_{\Omega_j} - n|^2 = 0$$

imply that Ω is a finite union of unit balls and that $P(\Omega_j) \to P(\Omega)$.

This second compactness statement is a particular case of a more general compactness result, related to the anisotropic version of Alexandrov's theorem. Define a *geometric integrand* to be a convex, one-homogenous function $F : \mathbb{R}^{n+1} \to [0,\infty)$, positive on the sphere. The *Wulff shape of F* is the bounded open convex set W_F,

$$W_F = \bigcap_{\nu\in\mathbb{S}^n} \Big\{x \in \mathbb{R}^{n+1} : x\cdot\nu < F(\nu)\Big\}\,.$$

The isoperimetric inequality (1) holds with W_F in place of B_1, and with the anisotropic energy

$$\mathcal{F}(\Omega) = \int_{\partial^*\Omega} F(\nu_\Omega)\, d\mathcal{H}^n$$

in place of $P(\Omega)$. In particular, *among SFP with fixed volume, F-Wulff shapes are the unique minimizers of $\mathcal{F}$* [7]. Given a variation $f_t(x) = x + t\,X(x) + O(t^2)$, the convexity of F implies the existence of the first variation

$$\delta\mathcal{F}|_\Omega(X) = \lim_{t\to 0^+} \mathcal{F}(f_t(\Omega))\,.$$

If Ω is a local minimizer of $\mathcal{F}$ at fixed volume, then $\delta\mathcal{F}|_\Omega(X) \geq 0$ on every $X \in C^1_c$ s.t. $\int_{\partial^*\Omega} X\cdot\nu_\Omega = 0$. A set of finite perimeter satisfying this property is *a critical point of $\mathcal{F}$ at fixed volume.*

Conjecture: *F-Wulff shapes are the unique sets of finite perimeter and finite volume that are critical points of $\mathcal{F}$ at fixed volume.*

This anisotropic version of Alexandrov's theorem is open even when Ω is assumed to be an open set with Lipschitz boundary rather than to be merely of finite perimeter. To the best of our knowledge this question seems to have been considered for the first time in a paper of Morgan [6], where it is affirmatively solved in the planar case, and, actually, in the most general case of immersed closed rectifiable curves. When F is smooth and λ-uniformly elliptic (i.e., $\lambda\,\mathrm{Id} \leq \nabla^2 F(\nu) \leq \mathrm{Id}/\lambda$ on $\nu^\perp$ for every ν), and Ω has a C^2-boundary, then the condition of being a critical point of $\mathcal{F}$ at fixed volume translates into

$$H^F_\Omega = \mathrm{div}^{\partial\Omega}(\nabla F(\nu_\Omega))$$

being constant. (By construction, we always have $H^F_{W_F} = n$.) Assuming that Ω is bounded, as proved by He, Li, Ma and Ge [5], H^F_Ω is constant if and only if Ω is an F-Wulff shape. From the physical viewpoint, the most significant case would however be that of *crystalline* integrands F, obtained as maxima of finitely many linear functions. In the joint paper [4] with Delgadino, Mihaila and Neumayer, we proved the following result, pointing to the validity of the above conjecture for *every* integrand F.

Compactness III: *If $\{F_j\}_j$ is a sequence of smooth, λ_j-elliptic integrands with $m \leq F_j \leq M$ for uniformly-in-j positive constants m and M; and if $\{\Omega_j\}_j$ is a sequence of open sets with smooth boundary and finite volume, normalized by scaling so to have $n\,\mathcal{F}_j(\Omega_j)/[(n+1)|\Omega_j|] = n$ and such that for a constant $\kappa > 0$*

$$H^{F_j}_{\Omega_j} \geq \kappa \text{ on } \partial\Omega_j\,; \tag{5}$$

then

$$F_j \to F \text{ on } \mathbb{R}^{n+1} \qquad \Omega_j \to \Omega \text{ in } L^1 \qquad \lim_{j\to\infty}\frac{1}{\lambda_j^2}\int_{\partial\Omega_j}|H^{F_j}_{\Omega_j} - n|^2 = 0 \tag{6}$$

imply that Ω is a finite union of F-Wulff shapes, with $\mathcal{F}_j(\Omega_j) \to \mathcal{F}(\Omega)$.

The conjecture would follow from Compactness III by solving the following:

Approximation problem: *Given a geometric integrand F, consider a bounded open set Ω with boundary at most as regular as that of W_F, and which is a critical point of $\mathcal{F}$ at fixed volume. Construct sequences $\{F_j\}_j$ of smooth λ_j-elliptic integrands, and $\{\Omega_j\}_j$ of bounded smooth sets satisfying* (5) *for some uniform $\kappa > 0$, in such a way that* (6) *holds.*

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Minimal surfaces and the Allen–Cahn equation on 3-manifolds

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(joint work with Otis Chodosh)

Fix (M^3, g) to be a closed Riemannian 3-manifold. The Allen–Cahn equation

$$\varepsilon^2 \Delta_g u = W'(u) \tag{1}$$

is a semilinear PDE which is deeply linked to the theory of minimal hypersurfaces. For instance, it is known that the Allen–Cahn functional

$$E_\varepsilon[u] := \int_M \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{W(u)}{\varepsilon} \right) d\mu_g,$$

whose critical points satisfy (1), Γ-converges as $\varepsilon \to 0$ to the perimeter functional [12, 14] and the level sets of E_ε-minimizing solutions to (1) converge as $\varepsilon \to 0$ to area-minimizing boundaries. When u is not E_ε-minimizing, the limit may occur with high multiplicity. Together with Otis Chodosh we studied solutions to (1) on 3-manifolds with uniform E_ε-bounds and uniform Morse index bounds and showed that multiplicity does *not* occur when the metric g is "bumpy," i.e., when no immersed minimal surface carries nontrivial Jacobi fields; bumpy metrics are generic in the sense of Baire category—see White [16]. This resolves a strong form of the "multiplicity one" conjecture of Marques–Neves [10] for Allen–Cahn. Our main theorem is:

Theorem 1 ([1]). *Suppose that u_i are critical points of E_{ε_i} with $\varepsilon_i \to 0$ and*

$$E_{\varepsilon_i}[u_i] \leq E_0, \ \operatorname{ind}(u_i) \leq I_0 \ \textit{for all } i = 1, 2, \ldots$$

Passing to a subsequence, for each $t \in (-1, 1)$, $\{u_i = t\}$ converges in the Hausdorff sense and in $C^{2,\alpha}_{\mathrm{loc}}$ away from $\leq I_0$ points to a smooth closed minimal surface Σ. For any connected component $\Sigma' \subset \Sigma$, either:

- *Σ' is two-sided and occurs as a multiplicity one graphical $C^{2,\alpha}$ limit; or,*