# ALEXANDROV'S THEOREM REVISITED

FRANCESCO MAGGI

UNIVERSITY OF TEXAS AT AUSTIN

JOINT WORK WITH MATIAS DELGADINO @ IMPERIAL COLLEGE

# ALEXANDROV'S THEOREM 1962

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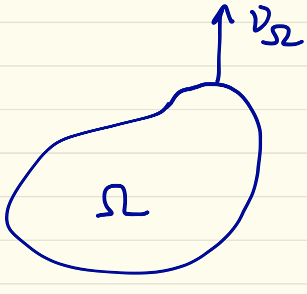
$\Omega \subseteq \mathbb{R}^{n+1}$ OPEN BOUNDED $C^2$ BOUNDARY CONNECTED



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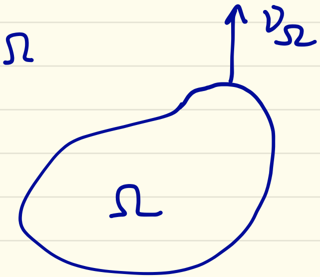

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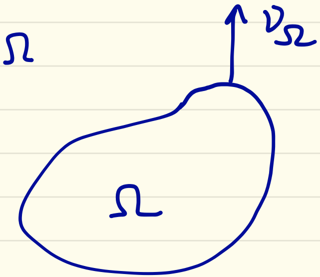

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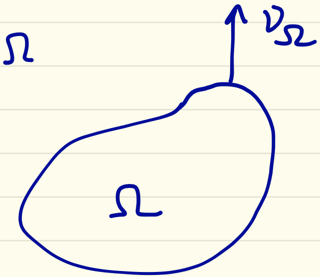


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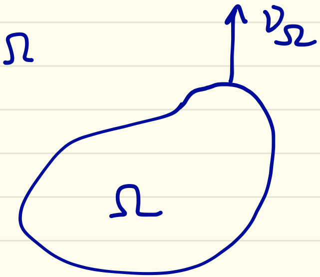


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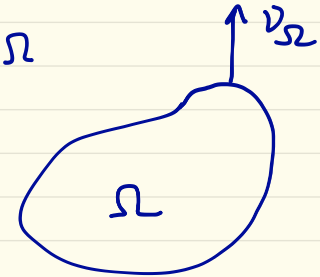


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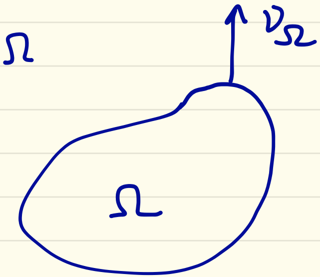


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**RMK** RELATED TO THE ISOPERIMETRIC THEOREM

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**HOW BAD CAN IT BE?**

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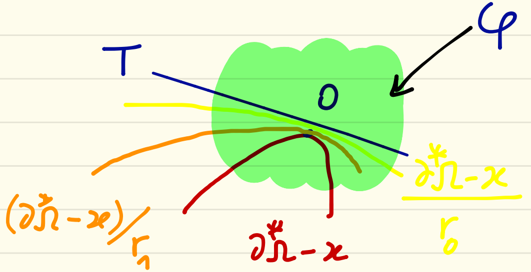

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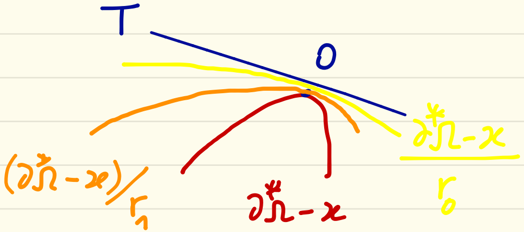


- THE DIVERGENCE THM HOLDS

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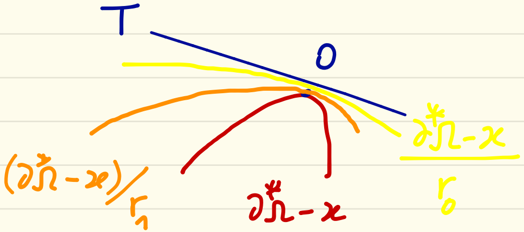


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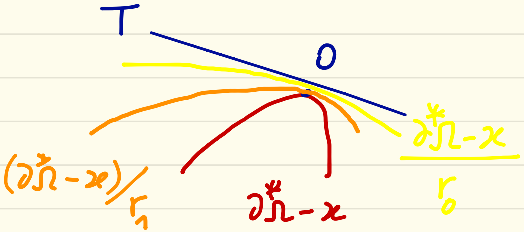


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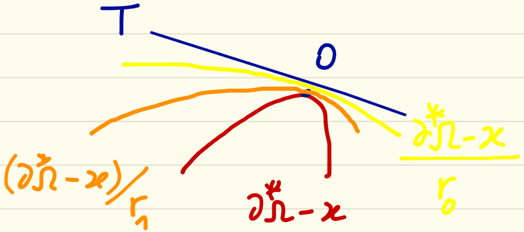


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WHERE $\begin{cases} f_t(x) = x + t\,X(x) + O(t^2) \\ X \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}) \end{cases}$ SMOOTH VARIATIONS

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**THUS** $\Omega$ VOL-CONSTRAINED CRITICAL POINT OF $P$

IFF $H_\Omega$ IS CONSTANT ON $\partial\Omega$

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[DELGADINO M. 2017] ALEX.'S THM REVISITED — FINITE UNIONS OF BALLS ONLY VOL-CONSTR. CRITICAL POINTS OF PERIMETER (AMONG CONNECTED SETS FINITE PER.)

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**RECALL** $\int_{\partial\Omega} \mathrm{div}^{\partial\Omega} X = \int_{\partial\Omega} H_\Omega (X \cdot \nu_\Omega)$ IF $\partial\Omega \in C^2$

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2015 $\partial\Omega \in C^2$ A FINITE UNION OF BALLS

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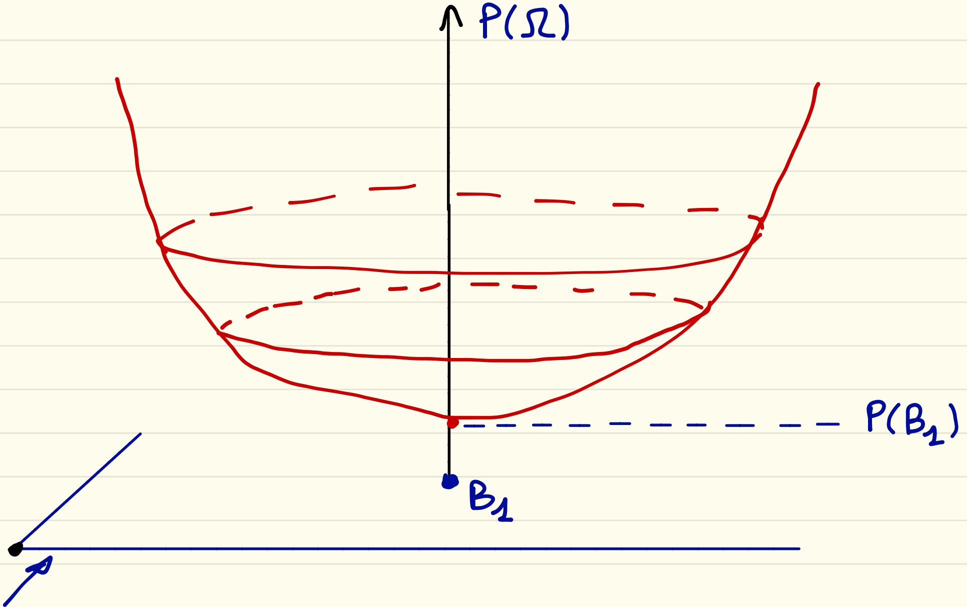
RMK BREZIS-CORON, STRUWE MID 80's FOR PARAMETRIZED CASE

# ISOPERIMETRIC LANDSCAPE (FIRST IMPRESSION)




CONNECTED SMOOTH SETS $\Omega$ WITH $H^{o}_{\Omega} = \dfrac{n\,P(\Omega)}{(n+1)\,|\Omega|} = n = H_{B_1}$ SCALING
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FINITE UNIONS OF TANG BALLS ARE
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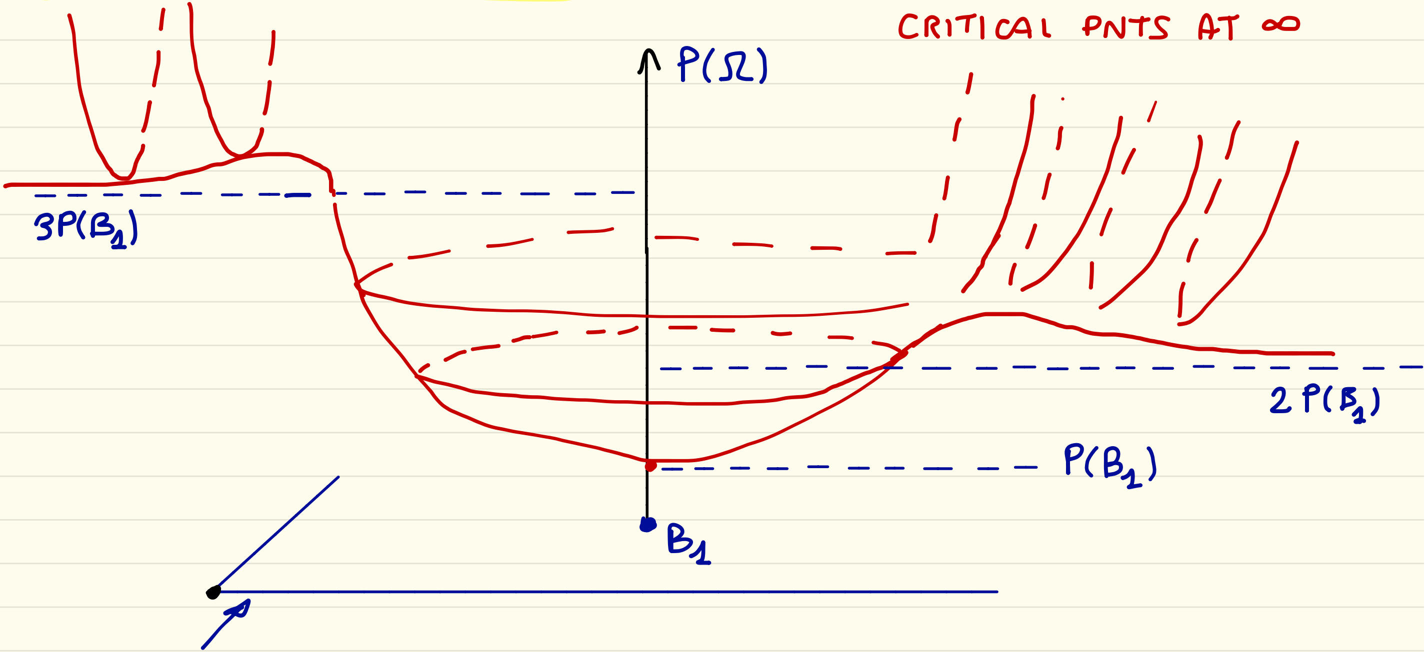


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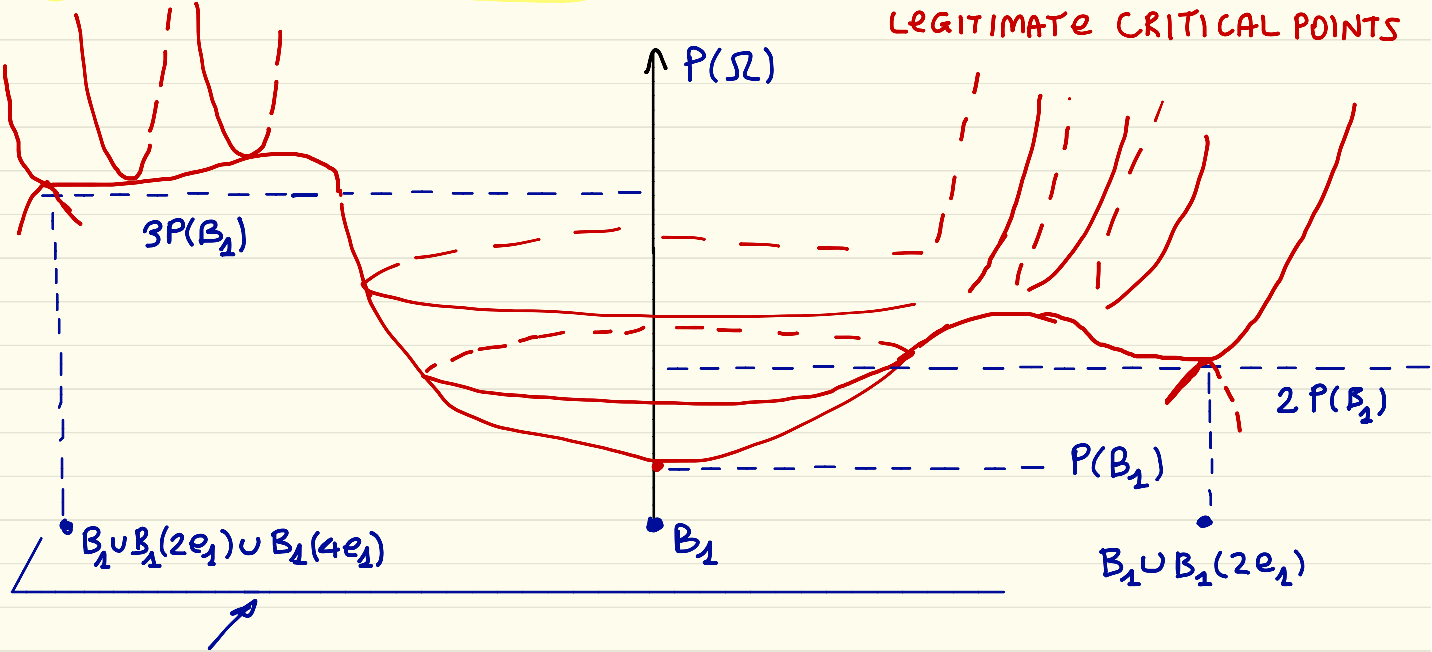
# ISOPERIMETRIC LANDSCAPE

FINITE UNIONS OF TANG BALLS ARE LEGITIMATE CRITICAL POINTS




CONNECTED SOFP $\Omega$ WITH $H^0_\Omega = \dfrac{n P(\Omega)}{(n+1)|\Omega|} = n = H_{B_1}$ SCALING

(MODULO RIGID MOTIONS)

# WHY BOTHER?

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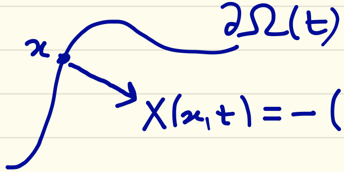
**VOLUME PRESERVING**
**MEAN CURVATURE FLOW**

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GRADIENT FLOW OF $P(\Omega)$ @ FIXED VOLUME

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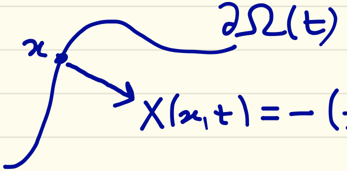


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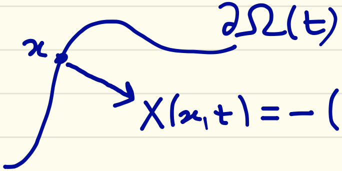


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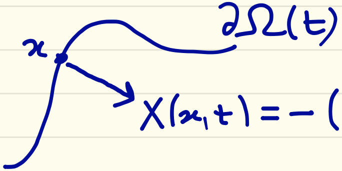


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$$= -\int_{\partial\Omega(t)} \left(H_{\Omega(t)} - \langle H_{\Omega(t)} \rangle\right)^2 \leq 0$$

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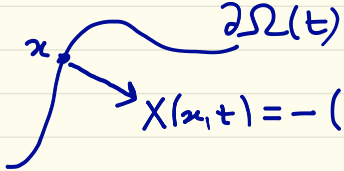


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**RMK** IF $\Omega$ SMOOTH UNIFORMLY CONVEX    HUISKEN

THEN GLOBAL-IN-TIME $\exists$ WITH $\Omega(t) \longrightarrow B^{|\Omega|}$ AS $t \to +\infty$

VPMCF - CONT.

RMK FOR GENERIC SMOOTH $\Omega$ FINITE TIME SINGULARITIES

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**BUT** REASONABLE GUESS $\Omega_\infty$ CMC SOFP

ALEXANDROV'S THM REV. (DELGADINO M.)

IF $\Omega$ S.T. $P(\Omega)<\infty$, $|\Omega|<\infty$,

$$\int_{\partial^*\Omega} \operatorname{div}^{\partial^*\Omega} X = \lambda \int_{\partial^*\Omega} X \cdot \nu_\Omega \quad \forall\, X \in C_c^\infty$$

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RMK IN VPMCF $\longleftrightarrow$ DISSIPATION INEQUALITY

$$\int_0^\infty dt \int_{\partial^*\Omega(t)} \left(H_{\Omega(t)} - \langle H_{\Omega(t)} \rangle\right)^2 \leq P(\Omega(0))$$

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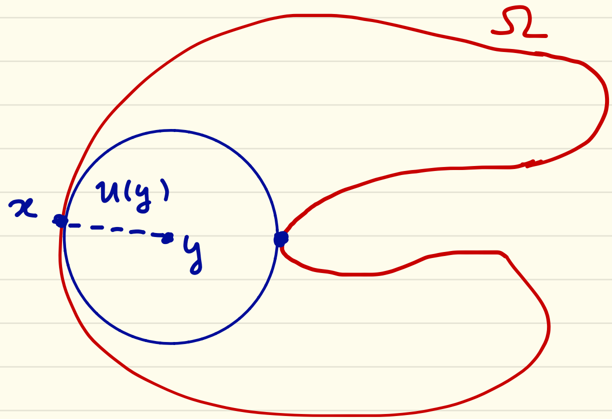

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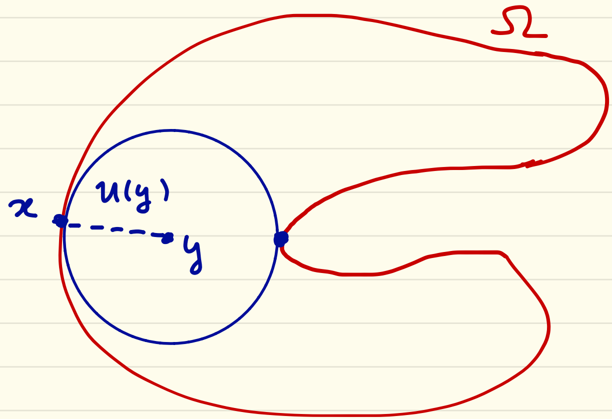

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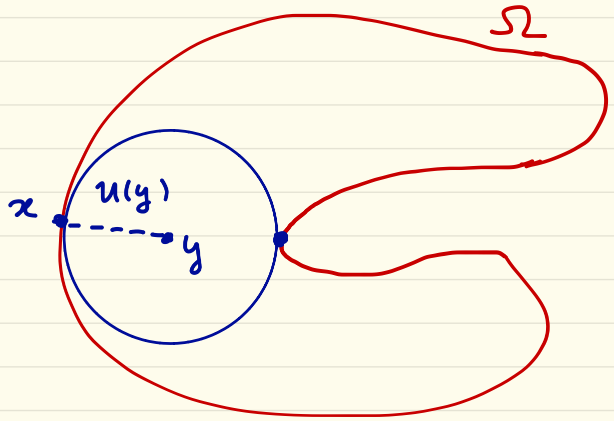

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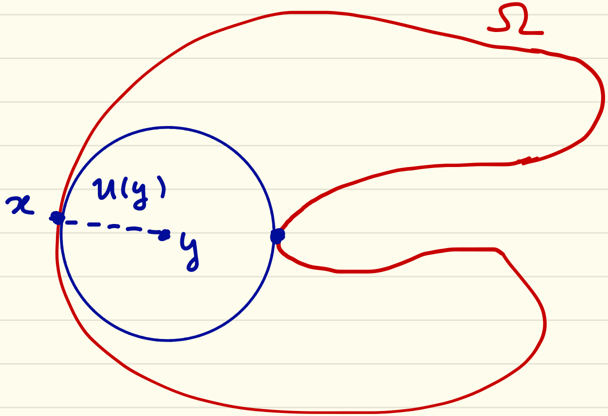

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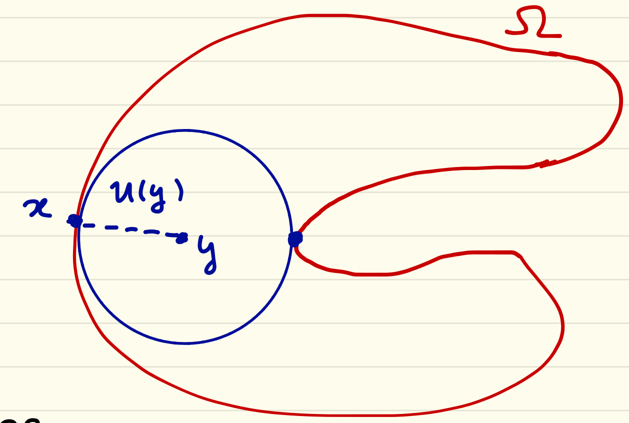


$$K_{max}(x) > \frac{1}{u(y)}$$

BY COMPARISON

IF $x$ **TOUCHING POINT** FOR $y$

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**RMK** **IF** $H_\Omega \equiv \lambda$ **THEN** $\lambda = H_\Omega^0 = nP(\Omega)/(n+1)|\Omega|$

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$$\le \int_{\partial\Omega} d\mathcal{H}^n \int_0^{n/H_\Omega} \left(1 - t \frac{H_\Omega}{n}\right)^n dt = \frac{n}{n+1} \int_{\partial\Omega} \frac{1}{H_\Omega}$$

**RMK** **IF** $H_\Omega \equiv \lambda$ **THEN** $\lambda = H_\Omega^o = nP(\Omega)/(n+1)|\Omega|$

$\Rightarrow$ HEINTZE KARCHER HOLDS =

**MONTIEL-ROS** $k_1 \le k_2 \le \dots \le k_n : \partial\Omega \to \mathbb{R}$ & $H_\Omega = \sum^n k_j > 0$

$Z = \{(x,t) : x \in \partial\Omega,\ 0 < t < k_n(x)^{-1}\}$

$\psi(x,t) = x - t\,\nu_\Omega(x) : Z \longrightarrow \psi(Z) \supseteq \Omega$

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NOW $\Omega$ SOFP WITH $\int_{\partial^*\Omega} \operatorname{div}^{\partial^*\Omega} X = \lambda \int_{\partial^*\Omega} X \cdot \nu_\Omega \quad \forall X \in C_c^\infty$

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$$\partial^*\Omega = \left\{x \in \partial\Omega : \omega_n = \lim_{\rho \searrow 0} \frac{\mathcal{H}^n(B_\rho(x) \cap \partial\Omega)}{\omega_n \rho^n}\right\}$$

$\partial^*\Omega$ OPEN IN $\partial\Omega$ ANALYTIC CMC

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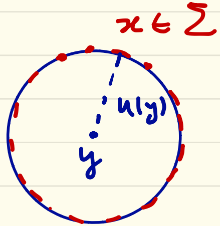


$\Sigma$ CLOSED WITH $\mathcal{H}^n(\Sigma) = 0$

**RMK** $Z = \{(x,t) : x \in \partial^*\Omega,\ 0 < t < k_n(x)^{-1}\}$

$\psi(x,t) = x - t\,\nu_\Omega(x)$

$y \in \Omega$ MAY HAVE ALL TOUCHING POINTS $x \in \Sigma$

RMK IF $x$ T.PNT FOR $y$ THEN $\exists\, \theta(x) \in \mathbb{N}$ & $T$ $n$-PLANE

S.T. $T_x(\partial\Omega) = \theta(x)\, T$

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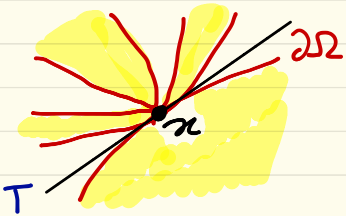


$\theta(x) = 5$

S.T. $T_x(\partial\Omega) = \theta(x)\, T$

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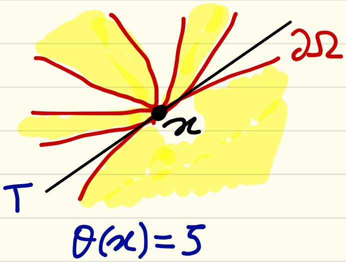


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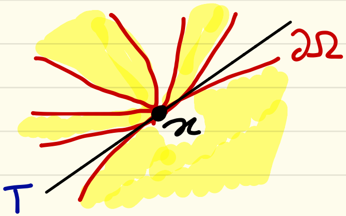
FOLIATION BY $C^{1,1}$ RECTIFIABLE CMPCT SETS $\Gamma_s^t$

RMK IF $x$ T.PNT FOR $y$ THEN $\exists\, \theta(x) \in \mathbb{N}$ & $T$ $n$-PLANE



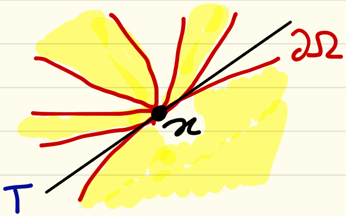

$\theta(x) = 3$

S.T. $T_x(\partial\Omega) = \theta(x)\, T$

FOLIATION BY $C^{1,1}$ RECTIFIABLE CMPCT SETS $\Gamma_s^t$

$\Omega_s = \{u = s\}$ $\quad u(y) = dist(y, \partial\Omega)$

RMK IF $x$ T.PNT FOR $y$ THEN $\exists\ \theta(x) \in \mathbb{N}$ & $T$ n-PLANE

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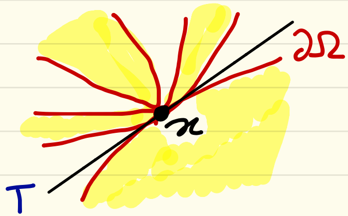
$\Gamma_s^t = \{ y \in \partial\Omega_s : \exists\ z \in \partial\Omega_t$ S.T. $|y - z| = t - s \}$ $\quad 0 < s < t$

RMK IF $x$ T.PNT FOR $y$ THEN $\exists\ \theta(x) \in \mathbb{N}$ & $T$ n-PLANE

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S.T. $T_x(\partial\Omega) = \theta(x)\, T$

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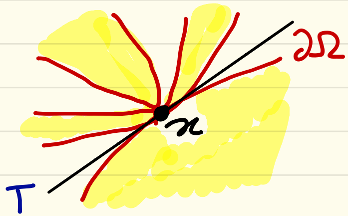
S.T. $T_x(\partial\Omega) = \theta(x)\, T$

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CLAIM 1 $\quad |\Omega \setminus \Omega^*| = 0$

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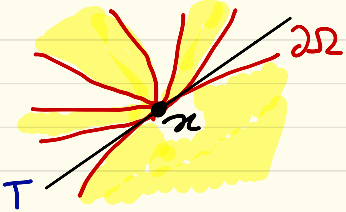


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CLAIM 1 $\quad |\Omega \setminus \Omega^*| = 0$ $\qquad$ CLAIM 2 $\quad |\Omega^* \setminus \psi(\Omega)| = 0.$

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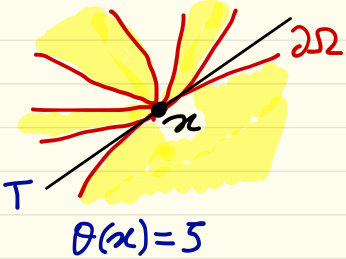


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CLAIM 1 $\quad |\Omega \setminus \Omega^*| = 0$ $\qquad$ CLAIM 2 $\quad |\Omega^* \setminus \psi(\Omega)| = 0.$

$\rightarrow$ CLAIM 2 IS DELICATE & EXPLOITS SCHÄTZLE'S MAXIMUM PRINCIPLE FOR VARIFOLDS

LUIS CAFFARELLI

1998

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# The Obstacle Problem Revisited

L.A. Caffarelli

The obstacle problem consists of studying the properties of minimizers of the Dirichlet integral

$$D(u) = \int_D (\nabla u)^2 \, dX$$

in a domain, $D$, of $R^n$, among all those configurations $u(X)$, with prescribed boundary values: $u|_{\partial D} = f(X)$, and constrained to remain, in $D$, above a prescribed obstacle $\varphi(X)$.

More precisely, we are given:

a) A (smooth) domain, $D$, of $R^n$.
b) A (smooth) function $f(X)$ on $\partial D$.
c) A (smooth) function $\varphi(X)$ on $D$, with $\varphi|_{\partial D} < f(X)$.

In the Hilbert space, $H^1(D)$, of all those functions with square integrable gradient, we define $K$ to be the closed convex set

$$K = \left\{ u \in H^1,\ u\Big|_{\partial D} = f(X),\ u \geq \varphi \right\} .$$

On $K$, there is a unique point $u_0$ that minimizes Dirichlet integral

$$D(u) = \int (\nabla u)^2 \, dX .$$

Such a point $u_0$ is called the "solution to the obstacle problem." Such a problem is motivated by the description of the equilibrium position of a membrane (the graph of $u$) that is "attached" at level $f(X)$ along the boundary of $D$, and is restricted to remain above $\varphi$, (the obstacle).

Such a membrane will minimize area integral

$$A(u) = \int \sqrt{1 + (\nabla u)^2} \, dX$$

that is linearized to Dirichlet integral for small deflections. In any case, the theory developed here applies to the "minimal surface," i.e., the non-linearized case, but for simplicity we will restrict our discussion here to the "linear" case.

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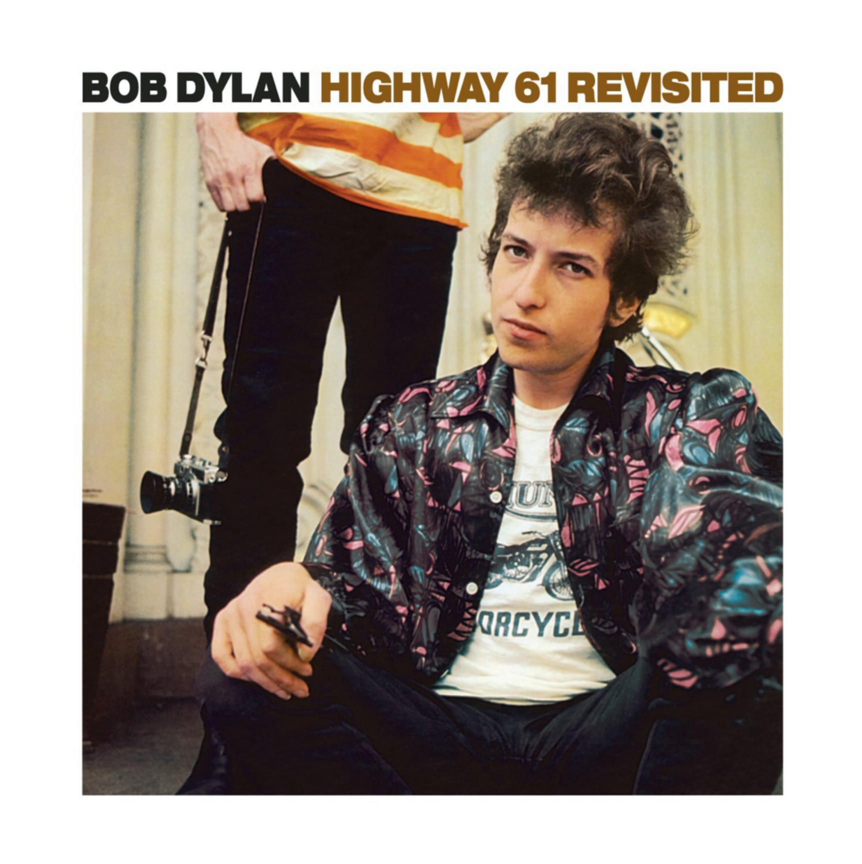


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