Remarks and errata to
"Sets of finite perimeter and geometric variational problems"
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Misprint 1. *Table of contents, page x:* Title of 21.3 misspelled "regularity". *(Federico Franceschini)*

Misprint 2. *Exercise 1.3, page 7:* $\mathrm{diam}(F_\varepsilon)$ should be $\mathrm{diam}(I_\varepsilon(F))$.

Misprint 3. *Proof of Theorem 2.8, page 18:* on the last line of step one and on line -5 from the end of step one: $\bigcup_{h\in\mathbb{N}} F_h \in \mathcal{F}$ and $\bigcap_{h\in\mathbb{N}} F_h \in \mathcal{F}$ should be changed, respectively, to $\bigcup_{h\in\mathbb{N}} F_h \in \mathcal{G}$ and $\bigcap_{h\in\mathbb{N}} F_h \in \mathcal{G}$. *(Tim Laux)*

Erratum 1. *Page 21, line 5:* Replace the whole line 5 with this:

"Then F is contained in the Borel set $J = H \cup (\mathbb{R}^n \setminus G)$. We conclude the proof by showing that $\mu(F) = \mu(J)$. This is obvious if $\mu(F) = \infty$, while, if $\mu(F) < \infty$, we have $\mu(F \cap (G \setminus E)) = 0$ and thus $\mu(F \cap G) = \mu(E \cap G)$: in particular"

Then the proof is concluded as in lines 6 and 7 from the book. *(Irvin Glick)*

Misprint 4. *Proposition 3.2, page 25, third line of the proof:* $\mathcal{F}$ should be $\mathcal{F}_\delta$. *(Filippo Cagnetti)*

Misprint 5. *Page 26, line 8:* "$g = f$ on E" missing in what should be "such that $g = f$ on E and $\mathrm{Lip}(g) = \mathrm{Lip}(f; E)$." *(David Simmons)*

Erratum 2. *Theorem 4.3, page 32–33:* starting at line -3 on page 32, it should be: ...and introduce a disjoint family of open sets $\{A_h^k\}_{h=1}^{N(k)}$ with $K_h^k \subset A_h^k \subset B_R$. Let $\varphi_h^k \in C_c^0(A_h^k; [0,1])$ with $\varphi_h^k = 1$ on K_h^k and define $u_k : \mathbb{R}^n \to \mathbb{R}$ by setting... and then it is ok starting from the formula for u_k at line 5, page 33.

Misprint 6. *Remark 4.9, page 35:* By Riesz's theorem, $\langle L, \varphi\rangle = \int_{\mathbb{R}^n} g\varphi \, d|L|$. *(Paul Rothnie)*

Erratum 3. *Exercise 4.13, page 36:* Identity $|f \cdot \mu| = |f|\,|\mu|$ should be replaced with the inequality $|f \cdot \mu| \le |f|\,|\mu|$. Equality always holds if $m = 1$, but may fail on specific f's if $m \ge 2$. For example, if $m = 2$, you may always take $f = g^\perp$ and find $f \cdot \mu = 0$. *(Yaroslav Vergun)*

Remark 1. *Exercise 4.16, page 37:* $\langle L, \varphi\rangle \le 2\,\langle L, \psi\rangle$ can be replaced by $\langle L, \varphi\rangle \le \langle L, \psi\rangle$. *(YV)*

Remark 2. *Example 4.24, page 42:* The notion of Γ being a "smooth curve"(which is informally used in this example – things are made precise in Section 10.2) should include the assumption that $\gamma'(t) \ne 0$ for every $t \in (a, b)$. *(TL)*

Remark 3. *Page 45, in equation (4.35):* Equation (4.35) is presented as a rewrite of (4.6). Since C_c^0 is used in (4.6) one could have more consistently used C_c^0 also in (4.35). The use of C_c^∞ in (4.35) is however correct by density. *(DS)*

Misprint 7. *Figure 5.1, page 54:* $r_h > (2/3)r_k$ should be $r_h \ge (2/3)r_k$. *(FC)*

Remark 4. *Theorem 5.8, page 58:* It could have been helpful to give an example where the set $\nu \llcorner \mu$ and $Y = \{x \in \mathrm{spt}\mu : D_\mu^+\nu = +\infty\}$: this is obtained, for example, by taking $\mu = \Lambda^2$ and $\nu = \mathcal{H}^1 \llcorner \ell$ for $\ell = \{x \in \mathbb{R}^2 : x_2 = 0\}$. Notice that in this case $\mu(D_\mu^+\nu = +\infty) = 0$ (as proved in general in step three of the proof below), but of course $\nu(D_\mu^+\nu = +\infty) = \nu(\ell) = +\infty$.

Remark 5. *Proof of Theorem 5.8, page 58:* In order to reduce the proof of the theorem to that of (5.14) one also needs to know that $\nu(\{x \in \mathrm{spt}\mu : D^-_\mu \nu(x) < D^+_\mu \nu(x)\}) = 0$. This last fact is not noticed in the text, but can be obtained by the same argument used in step three to show that $\mu(\{x \in \mathrm{spt}\mu : D^-_\mu \nu(x) < D^+_\mu \nu(x)\}) = 0$.

Erratum 4. *Page 59, last line in step two:* it is $\nu(E) \le$ rather than $\nu(E) = \sum_{h\in\mathbb{N}} \nu(\overline{B}(x_h, r_h))$. *(FC)*

Erratum 5. *Page 64, Remark 6.2:* on the third line, one needs to replace $r > 0$ with $r \in (0, 2)$, say. *(FC)*

Remark 6. *Proof of Theorem 6.4, page 66:* One could directly choose a bounded open set A containing M at the beginning of step one, and use the covering $\mathcal{F}'$ in place of the covering $\mathcal{F}$ also in the first part of the argument.

Misprint 8. *Page 68, line 11:* "then is" should be replaced by "then f is" *(FC)*

Remark 7. *Page 75, line 8:* It is claimed here that "$\nabla g_h(\tau) = \nabla f(x + h\,\tau)$ for every $\tau \in \mathbb{R}^n$" however this can be misleading because this is actually an identity between distributional gradients, and not a pointwise identity between classical gradients. Making use of the notation $\tau_h u(x) = u(x + h\,\tau)$, the claim could have been more clearly formulated as: "$\nabla g_h = \tau_h(\nabla f)$ as distributional gradients".

Erratum 6. *Page 75, line 14: it should be: We claim that*

$$g(0) = 0\,, \qquad \nabla g = \nabla f(x) \ \textit{a.e. on } B\,.$$

Indeed, since $\nabla g_h \to \nabla f(x)$ in $L^1(B; \mathbb{R}^m \otimes \mathbb{R}^n)$ as $h \to 0$, we find

$$\begin{aligned} -\int_{\mathbb{R}^n} \varphi\, \nabla g &= \int_{\mathbb{R}^n} g \otimes \nabla\varphi = \lim_{k\to\infty} \int_{\mathbb{R}^n} g_{\bar{h}(k)} \otimes \nabla\varphi = -\lim_{k\to\infty} \int_{\mathbb{R}^n} \varphi\, \nabla g_{\bar{h}(k)} \\ &= -\nabla f(x) \int_{\mathbb{R}^n} \varphi\,, \qquad \forall \varphi \in C^\infty_c(B)\,. \end{aligned}$$

A few lines below, one should replace "$\int_{\mathbb{R}^n} g_0 \nabla\varphi = 0$" with "$\int_{\mathbb{R}^n} g_0 \otimes \nabla\varphi = 0$". (FC)

Erratum 7. *Page 76, the last line of the statement of Theorem 8.1, "and $\mathcal{H}^n \llcorner f(\mathbb{R}^n)$ is a Radon measure on $\mathbb{R}^m$" should be replaced by "and if f is proper, then $\mathcal{H}^n \llcorner f(\mathbb{R}^n)$ is a Radon measure on $\mathbb{R}^m$". (Irving Glick)*

Misprint 9. *Page 76, line -2:* $\mathcal{H}^k$ should be $\mathcal{H}^n$ *(FC)*

Erratum 8. *Page 77, line 4:* The argument below (8.2) is for non-negative *simple* functions. Non-negative Borel functions are then addressed by approximation. *(FC)*

Misprint 10. *Page 79, line 3:* should be "$\{w_i\}_{i\in I}$ is an orthonormal set in $\mathbb{R}^m$" (not a basis until we complete it in the next line). *(DS)*

Misprint 11. *Page 79, line -10:* compect should be compact *(FC)*

Erratum 9. *Page 83, line -7 to line -5:* Remove these three lines. Deduce (8.19) by a direct application of Remark 8.6, see in particular (8.9).

Erratum 10. *Page 92:* In step two of the proof of Theorem 9.3 the order in which the open set A and the covering $\{B(x_k,s_k)\}_{k=1}^N$ are chosen should be reversed: Let $\phi \in C^1_c(\mathbb{R}^n)$ be given. By compactness there exist finitely many open balls $\{B(x_k,s_k)\}_{k=1}^N$ with $x_k \in \partial E$ which cover $\mathrm{spt}\phi \cap \partial E$. Choose an open set A such that $\mathrm{spt}\phi \cap \partial E \subset A \subset\subset \bigcup_{k=1}^N B(x_k,s_k)$. We may first choose a partition of unity $\{\zeta_k\}_{k=1}^N$ with $\zeta_k \in C^1_c(B(x_k,s_k);[0,1])$ such that $\sum_{k=1}^N \zeta_k = 1$ on A and then choose $\zeta_0 \in C^1_c(E;[0,1])$ such that $\sum_{k=0}^N \zeta_k = 1$ on $E \cap A$. *(DS)*

Erratum 11. *Page 98:* In the definition of **regular Lipschitz image** one should add a fourth condition:

$$\text{(iv) there exists } \lambda > 0 \text{ such that } |f(x) - f(y)| \geq \lambda |x - y| \text{ for every } x, y \in E\,.$$

Theorem 10.1 (Decomposition of rectifiable sets) already shows that it is always possible to guarantee the validity of (iv) whenever we use the notion of regular Lipschitz image. Condition (iv) is natural (it completes the analogy between regular Lipschitz image/C^1-embedding of a surface) and is technically convenient in proving Lemma 10.4, when we come to discuss the application of dominated convergence. Precisely, the proof of Lemma 10.4 should be modified as follows:

(a) Replace "there exists $r_0 > 0$ and $L > 0$ such that $\mathrm{spt}u_r \subset B_L$ for every $r \in (0, r_0)$;" with "there exists $L > 0$ such that $\{u_r > 0\} \subset B_L$ for every $r > 0$;"

(b) Replace the seven lines "We are left to prove the existence of r_0 and L [...] and $r_0 = s_0/R$ to conclude" with the following:

"We are left to prove the existence of L as above. Indeed, if $w \in \{u_r > 0\}$, then $z + r\,w \in E$, so that condition (iv) in the definition of regular Lipschitz image gives

$$|f(z + r\,w) - f(z)| \geq \lambda\, r\, |w|\,,$$

while at the same time $(f(z + r\,w) - f(z))/r \in \mathrm{spt}\varphi \subset B_R$, so that

$$|f(z + r\,w) - f(z)| \leq R\,r\,.$$

Combining the two inequalities, we prove that if $w \in \{u_r > 0\}$, then $|w| \leq R/\lambda = L$."

Misprint 12. *Page 98, last line:* g should be g_h

Misprint 13. *Page 99, line 3:* Kirszbraum should be Kirszbraun

Erratum 12. *Page 100, the two lines after (10.10):* The argument in these two lines just says that $\mathrm{spt}\, u_r \cap B_{s_0/r} \subset B_{R/\lambda}$ for $r < s_0/R$, and this is not sufficient to conclude. One may conclude by requiring E to be compact (and not only bounded). In this way, by compactness of E and by injectivity of f on E one has

$$\inf\{|f(z') - f(z)| : z' \in E \setminus B_{z,s_0}\} = \varepsilon_0 > 0\,,$$

so that, by (10.9) (that is, $|f(z') - f(z)| \geq \lambda |z - z'|$ for $z' \in B_{z,s_0}$) one gets

$$|f(z') - f(z)| \geq \min\Big\{\lambda, \frac{\varepsilon_0}{\mathrm{diam}(E)}\Big\}\, |z - z'| = c_0\, |z - z'|\,, \qquad \forall z' \in E\,.$$

In this way if $w \in \mathrm{spt}u_r$ then $x + r\,w \in E$ and $|f(z + rw) - f(z)| \leq R\,r$, so that $c_0\, r\, |w| \leq R\,r$. This proves $\mathrm{spt}u_r \subset B_{R/c_0}$, which is the claimed property. Notice that, by regularity of the Lebesgue measure, one can assume E to be compact in the definition of regular Lipschitz image on page 98 without affecting Theorem 10.1 and then its later use in Lemma 10.4 and Theorem 10.2.

Misprint 14. *page 107, line -4:* $P_1 = \sum_{h=1}^k w_h \otimes v_h$ should be replaced with $P_1 = \sum_{h=1}^k v_h \otimes w_h$ *(Qinfeng Li)*

Erratum 13. *Exercise 12.8, line 5,page 124:* $\mu_{x+\lambda E} = \Phi_\# \mu_E$ should be $\mu_{x+\lambda E} = \lambda^{n-1} \Phi_\# \mu_E$. *(Kenneth DeMason, DS)*

Erratum 14. *Exercise 12.11, page 124:* Formula (1.25) should be $\mu_{Q(E)}(F) = Q\,\mu_E(Q^* F)$ for every bounded Borel set $F \subset \mathbb{R}^n$. *(Felipe Hernandez)*

Erratum 15. *Proposition 12.13, page 125:* Line 4 of the proof, it should be $u(x) = -\mu_E((-\infty, x))$. Correspondingly, the chain of identities on line 8 leads to $\int_{\mathbb{R}} u\,\varphi' = \int_E \varphi'$, that in turn implies $\int_{\mathbb{R}} (u - 1_E)\,\varphi' = 0$ for every $\varphi \in C^1_c(\mathbb{R})$, and, finally, $u - 1_E = c$ a.e. on $\mathbb{R}$ for a suitable constant $c \in \mathbb{R}$. *(YV)*

Erratum 16. *Proposition 12.17, page 127:* "then either $|A \setminus E| = 0$ or $|E \cap A| = |A|$" should be replaced with "then either $|A \setminus E| = 0$ or $|E \cap A| = 0$". *(FH)*

Misprint 15. *Lemma 12.22, page 131:* it should be $A_k = \{x \in A \cap B_k : \mathrm{dist}(x, \partial A) > k^{-1}\}$ *(FH)*

Remark 8. *Example 12.15, page 131:* After "given $\{x_h\}_{h\in\mathbb{N}}$ dense in B and $\{r_h\}_{h\in\mathbb{N}} \subset (0, \varepsilon)$ such that $n\omega_n \sum_{h\in\mathbb{N}} r_h^{n-1} \le 1$" add "and $r_h < 1 - |x_h|$,". The latter condition is needed to guarantee that $E \subset B$. *(Rupert Frank)*

Misprint 16. *Page 132, line 2:* Replace $E_N = \sum_{h=1}^N B_h$ with $E_N = \bigcup_{h=1}^N B_h$. *(DS)*

Misprint 17. *Page 134, line -8:* w_n should be ω_n *(FC)*

Misprint 18. *Proposition 12.29, page 137:* in the statement of the propositions, it should ... for every F such that $F \setminus A = E_0 \setminus A$. *(FH)*

Misprint 19. *Page 138, line -13:* **sets** should be **set** *(FC)*

Misprint 20. *Proposition 12.31, page 140:* in the proof, first inequality, second integral, the domain of integration should be $E_h \cap F$ instead of $E \cap F$. *(FH)*

Misprint 21. *Exercise 12.33, page 140:* admit should be admits

Remark 9. *Proposition 12.37, page 143:* There is a quicker and more direct way to prove Proposition 12.37, which states the existence of a positive constant $c(n,t)$ depending on $n \ge 2$ and $t \in (0,1)$ such that

$$P(E; B_r) \ge c(n,t)|E \cap B_r|^{(n-1)/n}\,,$$

whenever $r > 0$ and $|E \cap B_r| \le t\,|B_r|$. This more direct argument is based on the Euclidean isoperimetric inequality from Chapter 14, and it goes as follows. Set $p(r) = P(E; B_r)$ and $v(r) = |E \cap B_r|$. By Equation (12.26) one has

$$n\omega_n^{1/n} v(r)^{(n-1)/n} \le P(E \cap B_r) \le p(r) + \mathcal{H}^{n-1}(E \cap \partial B_r)\,,$$
$$n\omega_n^{1/n}(\omega_n r^n - v(r))^{(n-1)/n} \le P(B_r \setminus E) \le p(r) + \mathcal{H}^{n-1}((\partial B_r) \setminus E)\,,$$

for every $r > 0$. Adding up

$$n\omega_n r^{n-1}\Big(\Big(\frac{v(r)}{\omega_n\,r^n}\Big)^{(n-1)/n} + \Big(1 - \frac{v(r)}{\omega_n\,r^n}\Big)^{(n-1)/n}\Big) \le 2p(r) + n\omega_n r^{n-1}\,,$$

that is, setting $\Psi(s) = s^{(n-1)/n} + (1-s)^{(n-1)/n} - 1$, $0 \le s \le 1$, we have

$$n\omega_n r^{n-1} \Psi\Big(\frac{v(r)}{\omega_n\, r^n}\Big) \le 2p(r)\,.$$

Since $\Psi(s) \ge \kappa(n,t)\, s^{(n-1)/n}$ for every $s \le t$ we find that

$$2p(r) \ge n\omega_n r^{n-1}\, \kappa(n,t)\, \Big(\frac{v(r)}{\omega_n\, r^n}\Big)^{(n-1)/n} \ge \kappa_0(n,t) v(r)^{(n-1)/n}\,.$$

Notice that possible values for $\kappa(n,t)$ are easy to compute, so that this argument allows one to obtain an explicit value for $c(n,t)$.

Remark 10. *Theorem 13.1 (Coarea formula), page 147:* The proof presented in the text (pages 148–150) works *verbatim* if the assumption "$u : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function" is replaced by "$u \in L^1_{\rm loc}(\mathbb{R}^n)$ has a distributional gradient $\nabla u \in L^1(\mathbb{R}^n;\mathbb{R}^n)$".

Misprint 22. *Example 13.3, page 147:* ... deduce that, *if* $u : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz... *(YV)*

Misprint 23. *Example 13.4, page 147:* it should be $\{u > t\} = \mathbb{R}^n \setminus \overline{B}(x,t)$ *(FC)*

Misprint 24. *Page 148, line 9:* increasing should be decreasing. *(FC)*

Misprint 25. *Page 149, line -9:* $(\psi' \circ u)\, \nabla u = -\varepsilon^{-1}$... should be $(\psi' \circ u)\, \nabla u = \varepsilon^{-1}$.... *(KDM)*

Erratum 17. *Page 150, line 6 of Theorem 13.8:* "whenever $P(E;\partial F) = 0$" should be "whenever $P(E;\partial F) = 0$ and F is a bounded Borel subset of $\mathbb{R}^n$". *(KDM)*

Erratum 18. *Proof of Theorem 13.8, page 152:* In step one of the proof one needs to assume $P(E;\partial A) = 0$. This is needed on line -8 to deduce from $|\nabla u_h|\, d\mathcal{L}^n \overset{*}{\rightharpoonup} |\mu_E|$ that $P(E;A) = \lim_{h\to\infty} \int_A |\nabla u_h|$. The rest of the proof goes without modification, one should just keep in mind that the radii r_i chosen in step two are such that $P(E; B_{r_i}) =$ for every $i \in N$.

Misprint 26. *Remark 13.12, page 153:* missing 0 at $|E \setminus B_R| \to$ as $R \to \infty$ *(FH)*

Remark 11. *Remark 13.12, page 153:* The proof can be largely simplified (also avoiding reference to Lemma 15.12) by noticing that when $|E| < \infty$, the function $u_\varepsilon(x) = 1_E \star \rho_\varepsilon(x)$ used in the proof of Theorem 13.8 is such that $u_\varepsilon(x) \to 0$ as $|x| \to +\infty$ (indeed, $u_\varepsilon(x) \le C(n)\, \varepsilon^{-n}\, |E \cap B_\varepsilon(x)| \le C(n)\, \varepsilon^{-n}\, |E \setminus B_{|x|-\varepsilon}(0)| \to 0$ as $|x| \to +\infty$ with ε fixed); therefore, the sets $E^t_h = \{u_{\varepsilon_h} > t\}$ with $t > 0$ are automatically bounded. *(RF)*

Misprint 27. *Page 156, line 2:* The inclusion

$$(u \circ \psi)(F) \subset \Big\{ y \in \mathbb{R}^{n-1} : \nabla(u \circ \psi)(y) = 0 \Big\}$$

should be replaced by the inclusion

$$(u \circ \psi)(F) \subset\ (u \circ \psi)\bigg(\Big\{ y \in \mathbb{R}^{n-1} : \nabla(u \circ \psi)(y) = 0 \Big\} \bigg)$$

(TL)

Misprint 28. *Page 157, line 4:* $P(B) = \omega_n$ should be $P(B) = n\, \omega_n$. *(TL)*

Misprint 29. *page 160, line 3:* just notation, $P(E_z, I)$ should be $P(E_z; I)$ *(FH)*

Misprint 30. *page 162, line 20:* $\mathcal{H}^{n-1}(D_h)$ should be $\mathcal{H}^{n-1}(D_h)^2$ *(FH)*

Erratum 19. *page 162, line -6:* It is false that $G_h \to G$ in L^1. Indeed, when $G_h = \{m_h > 0\}$, $G = \{m > 0\}$ and $m_h \to m$ in L^1 it could still happen that $\liminf_h |G_h \setminus G| > 0$. To fix the problem, on line -4 replace $1_{G_h \setminus D_h}$ with $1_{G \setminus D_h}$, and correspondingly, on line -2 gives

$$\int_G P(E_z)\,dx \le \liminf_{h\to\infty} \int_{G\setminus D_h} P((E_h)_z)\,dx = 2\liminf_{h\to\infty} \mathcal{H}^{n-1}(G\setminus D_h) = 2\mathcal{H}^{n-1}(G)\,.$$

The rest of the proof continues in the same way. *(José Gomes, Serena Quagreda)*

Erratum 20. *page 163, line 7:* "pair of concave non-negative functions $\psi_1, \psi_2 : C \to [0,\infty)$" should be replaced by "pair of concave functions $\psi_1, \psi_2 : C \to \mathbb{R}$ with $\psi_1 + \psi_2 \ge 0$" *(KDM)*

Erratum 21. *page 163, lines 17–19:* $C \setminus \overline{C'}$ should be replaced (three instances) by $\mathbb{R}^n \setminus \overline{C'}$. *(Isaac Neal)*

Remark 12. *page 163, Lemma 14.6:* The basic measure-theoretic fact proved in this lemma (which, indeed, makes no use of the finite perimeter assumption stated in the lemma!) already appears in Gonzalez and Greco, *Una nuova dimostrazione della proprietà isoperimetrica dell'ipersfera nella classe degli insiemi aventi perimetro finito*, Ann. Univ. Ferrara - Sez. VII, Sc. Mat. VOL XXIII, 251–256, (1977). *(RF)*

Misprint 31. *page 168, last line of Example 15.4:* $|(e_1 \pm e_2)/2|$ in place of $|(e_1 + e_2)/2|$ *(KDM)*

Misprint 32. *page 168, 3 lines above Theorem 15.5:* "reduce boundary" should be "reduced boundary" *(Georgios Domazakis)*

Misprint 33. *page 169, equation (15.7):* , should be .

Erratum 22. *page 173, Exercise 15.13:* one needs to add the assumption $|E| < \infty$ in order to prove (15.23) – as it is correctly done in Proposition 19.22 *(Monica Torres)*

Misprint 34. *page 174, line -6:* the identity

$$-\int_{\mathbb{R}^n} \frac{\partial u_\varepsilon}{\partial \nu}\,\varphi = \int_{\mathbb{R}^n} \varphi\,d|\mu_E|\,,$$

should be replaced by

$$-\int_{\mathbb{R}^n} \frac{\partial u_\varepsilon}{\partial \nu}\,\varphi = \int_{\partial^* F} \varphi_\varepsilon\,d|\mu_F|\,, \qquad \text{where } \varphi_\varepsilon = \varphi \star \rho_\varepsilon\,.$$

Then one has to notice that if $\varphi \ge 0$ on $\mathbb{R}^n$, then $\varphi_\varepsilon \ge 0$ on $\mathbb{R}^n$ for every $\varepsilon > 0$.

Erratum 23. *page 177:* In the last two lines of page 177, all but one α should be replaced with $|\alpha|$. Precisely, the two lines should have been:
If $\alpha < 0$, then $F \subset H_x$ and $|F \cap B_{-\alpha}| = 0$, so that

$$0 = \frac{|F \cap B_{-\alpha}|}{|B_{-\alpha}|} = \lim_{h\to\infty} \frac{|E_{x,r_h} \cap B_{-\alpha}|}{|B_{-\alpha}|} = \lim_{h\to\infty} \frac{|E \cap B(x, -r_h\alpha)|}{|B(x, -r_h\alpha)|}\,,$$

(Francesco Ferraresso)

Erratum 24. *Example 16.13: Page 190, lines 17 and 18:* $\mu_E \llcorner A' = \mu_F \llcorner A'$ for some open set A' with $\mathbb{R}^n \setminus A \subset A'$ (since we need to choose A' with $E\Delta F \cap A' = \varnothing$ and $\partial A \subset A'$). *(DS)*

Misprint 35. *page 191, 2 lines below (16.29)* there are two $B(x_1 r)$ that should be $B(x, r)$. *(GD)*

Misprint 36. *Page 197, line -8:* in the right-hand side of the equation, φ should be composed with f instead of g. *(DS)*

Misprint 37. *page 198, last line:* the last summation is over $1 \le i < j \le n$ rather than on $1 \le i, j \le n$. *(QL)*

Misprint 38. *page 201, below (17.26):* "$\mathbf{H}_E$ defines a signed Radon measure" should be changed to "$\mathbf{H}_E$ defines an $\mathbb{R}^n$-valued Radon measure" *(TL)*

Misprint 39. *page 203, lines 3-4:* $\mathrm{O}(t^2)$ should be replaced by $\mathrm{o}(t)$ since g is just assumed of class C^1. *(TL)*

Misprint 40. *page 205, Remark 17.15:* $P(E_r) < P(E)$ should be replaced by "$P(E_r; B_R) < P(E; B_R)$ for every $R > r$". *(TL)*

Remark 13. *page 207, proof of Corollary 17.18:* In the displayed equation after (17.44), the second equality can be asserted only for a.e. $r > 0$; therefore, we have showed that if $x \in A \cap \partial E$, then for a.e. r such that $B(x,r) \subset A$, we have $P(E; B(x,r)) \ge \omega_{n-1}\, r^{n-1}$. Once this is proved, the monotonicity of $\gamma(x,r) = P(E; B(x,r))/\omega_{n-1}\, r^{n-1}$ for $x \in A$ and $r \in (0, \mathrm{dist}(x, \partial A)$ is enough to conclude the validity of $\gamma(x,r) \ge 1$ for every $x \in A \cap \partial E$ and $B(x,r) \subset A$. *(TL)*

Misprint 41. *page 212, line 17:* $(n-1)\Phi(r) - r\,\Phi'(r)$ should replace $(n-1)\Phi(r) - \Phi'(r)/r$. *(QL)*

Misprint 42. *page 212, line 2:* reference to Giusti's book is for Appendix B, not Appendix A. *(QL)*

Remark 14. *page 223, proof of Theorem 18.8:* One should be careful with the fact that letter t is used both as the "almost-flatness" parameter in the decomposition of M and as the level set parameter in the coarea formula: this is of course unintentional.

Misprint 43. *page 223, eqns (18.17) and (18.18):* f should be u *(QL)*

Remark 15. *page 224, proof of Theorem 18.8, line 3-4:* One should better say: If now $z \in E_h$ is such that $(u \circ g_h)$ is differentiable at z (as it happens to be the case for a.e. $z \in E_h$ by Rademacher's theorem), then by Lemma 11.5 u is tangentially differentiable at $g_h(z)$ with respect to M_h, with [formula on line 5].

Misprint 44. *section 19.2, starting at page 237:* Throughout this section, **Schwartz** should be **Schwarz**. *(RF)*

Misprint 45. *Page 240, line 13:* in the definition of $r(t)$ the power $n-1$ should be replaced by $1/(n-1)$. *(TL)*

Misprint 46. *page 241, equation (19.37):* Inside the integral replace $p_E^2 - p_{E^*}^2$ with $\sqrt{p_E^2 - p_{E^*}^2}$ (the integrand appears correctly elsewhere in the proof). *(RF)*

Misprint 47. *Page 241, line -9:* as on page 240: in the definition of $r(t)$ the power $n-1$ should be replaced by $1/(n-1)$. *(TL)*

Remark 16. *Page 244, Proposition 19.18:* Here it is implicitly assumed that $E^{(1/2)} \cap \partial H \neq \emptyset$ (see line 4 of step two). If we assume the convention that $\mathbf{D}_r = \emptyset$ when $r = 0$, then the statement of the proposition is correct by changing "there exists $r_0 > 0$" with "there exists $r_0 \ge 0$". *(TL)*

Misprint 48. *Page 246:* In the displayed equation two lines below (19.53), $B(t\,e_n, r)$ should be $B(s\,e_n, r)$. *(TL)*

Misprint 49. *page 249, line 9:* replace the equality with $\le$. *(RF, TL)*

Remark 17. *Theorem 19.23, page 250:* The assumption "$g \in L^1(\mathbb{R}^n)$" is evidently a left-over of an unfortunate copy and paste. The correct assumptions on g are that: $g = g(x_n)$ is Borel measurable and E is such that $\int_E g(x_n)\,dx < \infty$. Moreover, on the right-hand side of (19.62), $\mathcal{G}(F)$ should have appeared in place of $\mathcal{G}(E)$. *(RF)*

Misprint 50. *Page 253, line -7:* $\mu_{E_h} \to \mu_E$ should be $\mu_{E_h} \stackrel{*}{\rightharpoonup} \mu_E$. *(TL)*

Misprint 51. *Page 254, second displayed equation:* Two instances of ∂F should be $\partial^* F$. *(TL)*

Misprint 52. *Page 255:* in the second displayed equation, the factor g/M should just be $1/M$; in the third displayed equation, $2\gamma/M$ should be $2\gamma/g\,M$. *(TL)*

Misprint 53. *Page 282:* Title of 21.3, misspelled "regularity". *(FF)*

Misprint 54. *Pages 288, line 7:* In the proof of Theorem 21.14, it mentions "arguing as in step two" when the labeling in the steps of the proof on pages 286 and 287 jump from "step one" to "step three" without a "step two". *(DS)*

Misprint 55. *Exercise 22.7: Page 293, line -4:* $x \in \partial E$ is stated unnecessarily; also, it is repeated a second time in line -2. *(DS)*

Misprint 56. *page 302, equation (22.50):* the integral $\int_{t_0}^1$ should be $\int_{t_0}^{t_1}$. *(QL)*

Misprint 57. *page 304–305:* as formulated, the statement of Theorem 23.1 suggests that (23.3) and (23.4) correspond, respectively, to (23.6) and (23.7); while actually, (23.7) holds under (23.3), and (23.6) holds under (23.4). This mix-up is also present on lines 2 and 3 of "Step three" on page 305. *(DS)*

Misprint 58. *page 332, eqns (24.35) and (24.37):* $d\mathcal{H}^{n-2}$ to be replaced with $d\mathcal{H}^{n-1}$ *(QL)*

Misprint 59. *page 347:* "Proof of Theorem 26.3" is actually "Proof of Theorem 26.1" (Theorem 26.1 is the case $\Lambda = 0$ of Theorem 26.3). *(TL)*

Remark 18. *page 354, Equation (26.48):* Note that given $x \in A \cap \partial E$, the condition

$$\inf_{r\in(0,r_0),B_{x,r}\subset\subset A} \mathbf{e}(E,x,r) + \Lambda\,r \ge \varepsilon(n)$$

is trivially equivalent to

$$\inf_{r\in(0,r_0),B_{x,r}\subset\subset A} \mathbf{e}(E,x,r) \ge \varepsilon(n)\,.$$

Erratum 25. *page 365:* the displayed equation before (28.5) should be changed into

$$\partial F(r) \cap \partial B_s = \Big\{\frac{s}{r}\,x : x \in \partial F \cap \partial B_r\Big\}\,, \qquad \forall s > 0\,,$$

Erratum 26. *page 374, line -2,* $|\nabla\varphi| = u(|x|)$ should be replaced by $|\nabla\varphi| = |u'(|x|)|$.

Erratum 27. *page 388, line 21:* "$\omega(t)/t$ is increasing" should be replaced by "$\omega(t)/t$ is decreasing".

Erratum 28. *page 393, equation (1):* $2(12)^{1/4}$ should be $(12)^{1/4}$ *(Marco Caroccia)*

Misprint 60. *page 395, line 11:* "an suitable" should be "a suitable". *(TL)*

Erratum 29. *page 401:* Equation (29.16) holds true for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{E}(h,k)$ but not, in general, for every $x \in \mathcal{E}(h,k)$. (Indeed, think to the case when $\mathcal{E}(h)$ and $\mathcal{E}(k)$ are two tangent balls, $\mathcal{E}(j)$ is their complement, and x is the tangency point.) To show that equation (29.16) holds true for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{E}(h,k)$, one notices that $\theta_{n-1}(\partial^*\mathcal{E}(j))(x) = 0$ $\mathcal{H}^{n-1}$-a.e. on $\mathbb{R}^n \setminus \partial^*\mathcal{E}(j)$ – see Corollary 6.5 – as well as that, since $\mathcal{E}(h)$, $\mathcal{E}(k)$, and $\mathcal{E}(j)$ are disjoint (modulo $\mathcal{H}^n$) sets of finite perimeter in $\mathbb{R}^n$, then

$$\partial^*\mathcal{E}(h) \cap \partial^*\mathcal{E}(k) \cap \partial^*\mathcal{E}(j) = \emptyset.$$

We also notice that (29.16) is used in the proof of Lemma 29.13 (see the inequality after (29.50)). Correspondingly, the statement of Lemma 29.13 has to be changed into:

If $n \geq 2$, $\mathcal{E}$ is a cluster in $\mathbb{R}^n$, $0 \leq h < k \leq N$, $z \in \mathcal{E}(h,k)$ is such that $\theta_{n-1}(\partial^*\mathcal{E}(j))(z) = 0$ for every $j \neq h,k$, and $\delta > 0$, then there exist positive constants [...]

Accordingly, in the proof of Theorem 29.14, the points z_α and y_α has to be chosen by taking care of this modification. For example, the line before (29.53) should be changed into "each z_α is an interface point of $\mathcal{E}$ such that $z_\alpha \in \mathcal{E}(h(\alpha),k(\alpha))$, $\theta_{n-1}(\partial^*\mathcal{E}(j))(z_\alpha) = 0$ for every $j \neq h(\alpha), k(\alpha)$, and $\mathcal{H}^{n-1}(\mathcal{E}(h(\alpha),k(\alpha))) > 0$". *(Michele Marini and Guido De Philippis)*

Misprint 61. *Page 443, line 5:* $(x_k - x_0)/r_k \to v \in S^1$ should be replaced by $(x_k - x_0)/r_k = v \in S^1$.