Combinatorial Knot Heegaard Floer Homology

Allison Moore

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Introduction

Grid diagrams

Multipointed H.F.H.

Proof

Example

$\widehat{HFK}$

Given an oriented, nullhomologous knot K in a closed, oriented 3-manifold Y, one associates a sequence of bigraded abelian groups,

$$\widehat{HFK}(Y; K, s, m)$$

called the *knot Heegaard Floer homology groups of K in Y*.

- This graded object is an invariant of K.
- Categorifies the Alexander polynomial:

$$\Delta_K(T) = \sum_{m,s} (-1)^m \text{rank } \widehat{HFK}_m(Y; K, s) \cdot T^s$$

A brief history of Heegaard Floer homology

2001 Ozsváth and Szabó (OS) develop Heegaard Floer homology for closed, oriented 3-manifolds Y.

2002 OS and Jacob Rasmussen simultaneously develop knot Heegaard Floer homology.

Categorification of the the Alexander polynomial

2003-4 Progress continues. Some applications:

detect the genus of a knot.
distinguish some mutants.
slice genus and unknotting number.
Khovanov Homology.

2005 OS adapt to multipointed Heegaard Diagrams.

Multipointed H.F.H. generalizes the multivariable Alexander polynomial.
Knot H.F.H. detects fibered knots (Yi Ni).

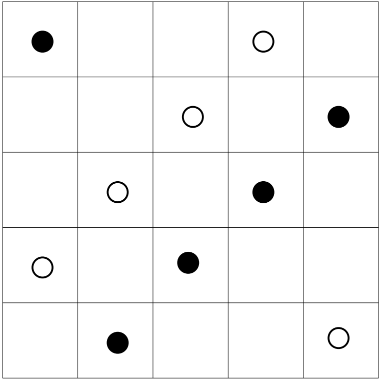
2006 Combinatorial methods.

Manolescu, Ozsváth, and Sarkar develop combinatorial methods for knots and links in S^3.

2007-9 Progress expands and continues.

Baker, Grigsby, and Hedden outline a program towards the Berge conjecture.
New theories: sutured, twisted, bordered H.F.H.

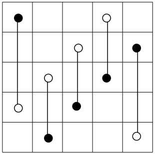
Grid diagram $\tilde{\Gamma}$



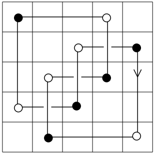
A grid diagram of grid number 5.
Every row containts exactly one black dot and white dot.
No two dots share a box.

A compatible oriented link L

- Vertical segments connect black dots to white dots.



- Horizontal segments connect white dots to black dots as underpasses.

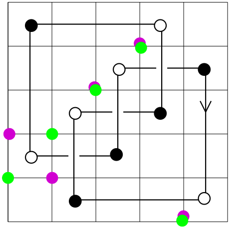


Transfer $\tilde{\Gamma}$ to a torus $\mathcal{T}$ to obtain the toroidal grid diagram Γ.

$(C(\Gamma), \partial)$

Definition: For Γ of grid number n, let $C(\Gamma)$ be the free abelian group generated by n-tuples of lattice points on the grid, where an n-tuple intersects each embedded circle exactly once.

- The set X of n-tuples corresponds with S_n.



- Let $\mathbf{x}_0$ denote the special n-tuple consisting of the lower left hand corner of the squares marked with white dots.

Alexander grading

- $a(p) = -\omega_K(p)$ is the winding number of the knot around a point.
- $c_{i,j}$ is the j-th corner of the i-th square containing a dot.
- **Definition:** Alexander grading $\mathcal{A} : C(\Gamma) \to \mathbb{Z}$

$$\mathcal{A}(\mathbf{x}) - \mathcal{A}(\mathbf{y}) = \sum_{x \in \mathbf{x}} a(x) - \sum_{y \in \mathbf{y}} a(y)$$

$$\mathcal{A}(\mathbf{x}) = \sum_{x \in \mathbf{x}} a(x) - \frac{1}{8} \sum_{i,j} a(c_{i,j}) - \frac{n-1}{2}$$

Domains

- Let D denote a 2-chain in the cellular homology of Γ with ∂D contained in the embedded circles of Γ.
- Let $\gamma_{\mathbf{x},\mathbf{y}}$ denote an oriented null-homologous curve supported on the embedded circles, where horizontal circles connect $\mathbf{x}$ to $\mathbf{y}$.
- Define $p_x(D)$ to be the average local multiplicty of D at x, and

$$P_{\mathbf{x}}(D) = \sum_{x \in \mathbf{x}} p_x(D)$$

- Let $B(D)$, $W(D)$ be the number of black and white dots in D.

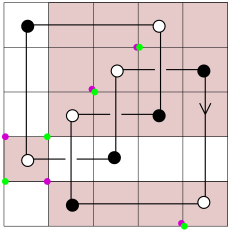
Maslov grading

Definition: Let D be a two chain such that $\partial D = \gamma_{\mathbf{x},\mathbf{y}}$. Then $\mathcal{M} : C(\Gamma) \to \mathbb{Z}$ is defined by

$$\begin{aligned} \mathcal{M}(\mathbf{x}) - \mathcal{M}(\mathbf{y}) &= P_{\mathbf{x}}(D) + P_{\mathbf{y}}(D) - 2 \cdot W(D) \\ \mathcal{M}(\mathbf{x}_0) &= 1 - n \end{aligned}$$

Rectangles

An embedded rectangle r is said to connect $\mathbf{x}$ to $\mathbf{y}$ if $\mathbf{x}$ and $\mathbf{y}$ agree on all but four lattice points, and if $\partial r = \gamma_{\mathbf{x},\mathbf{y}}$.



$R_{\mathbf{x},\mathbf{y}}$ is the set of rectangles r connecting $\mathbf{x}$ to $\mathbf{y}$.
$\#R_{\mathbf{x},\mathbf{y}} = 2$ or 0.

$\partial : C(\Gamma) \to C(\Gamma)$

$C(\Gamma)$ is endowed with the differential

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in X} \sum_{r \in R_{\mathbf{x},\mathbf{y}}} c(r) \cdot \mathbf{y}$$

Where

$$c(r) = \left\{ \begin{array}{ll} 1 & \text{if } P_{\mathbf{x}}(r) + P_{\mathbf{y}}(r) = 1; W(r) = B(r) = 0 \\ 0 & \text{otherwise} \end{array} \right\}$$

- $\partial^2 = 0$.
- ∂ drops $\mathcal{M}$ by 1.
- ∂ preserves $\mathcal{A}$.

Theorem (Manolescu, Ozsváth, Sarkar, 2006): Fix a grid presentation Γ of K, with grid number n. Then,

$$H_*(C(\Gamma), \partial) \cong \widehat{HFK}(K) \otimes V^{\otimes(n-1)}$$

Where V is a 2-dimensional bigraded vector space, with one generator in bigrading $(-1, -1)$ and one generator in bigrading $(0, 0)$.

Proof Sketch

Step 1 **Proposition (MOS/OS; 2006):** Let $(\Sigma_g, \alpha, \beta, \mathbf{w}, \mathbf{z})$ be a $2k$-pointed Heegaard Diagram compatible with K. Then

$$\widehat{HFK}(K) \otimes V^{\otimes(k-1)} \cong H_*(C(\Sigma_g, \alpha, \beta, \mathbf{w}, \mathbf{z}), \partial)$$

Step 2 Identify $C(\Gamma)$ with $C(\Sigma_g, \alpha, \beta, \mathbf{w}, \mathbf{z})$.

Step 3 Show that the Alexander gradings A and $\mathcal{A}$ agree.

Step 4 Show that the Maslov gradings M and $\mathcal{M}$ agree.

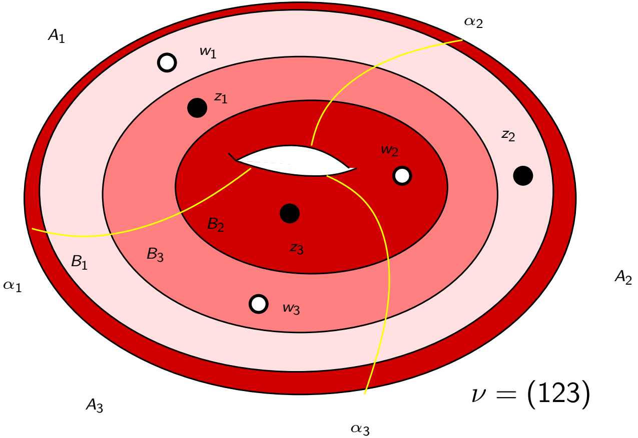
Step 5 Show the differentials agree.

Heegaard diagrams

- Let U_α and U_β be genus g handlebodies with common boundary Σ_g.
- $\alpha = \{\alpha_1, \cdots, \alpha_{g+k-1}\}$ and $\beta = \{\beta_1, \cdots, \beta_{g+k-1}\}$ are sets of attaching circles for U_α and U_β.
- $\mathbf{w} = \{w_1, \cdots, w_k\}$ and $\mathbf{z} = \{z_1 \cdots, z_k\} \subset \Sigma_g - \alpha - \beta$ are basepoints.
- Denote by $\{A_i\}_{i=1}^k$ and $\{B_i\}_{i=1}^k$ the connected components of $\Sigma_g - \alpha$ and $\Sigma_g - \beta$.

We say $(\Sigma_g, \alpha, \beta, \mathbf{w}, \mathbf{z})$ is a $2k$-pointed Heegaard diagram for Y compatible with K.

Heegaard Diagram



$\nu = (123)$

A_i contains z_i and w_i.

B_i contains w_i and $z_{\nu(i)}$ for some permutation ν.

A compatible link L

ξ_i is an arc in A_i from z_i to w_i.

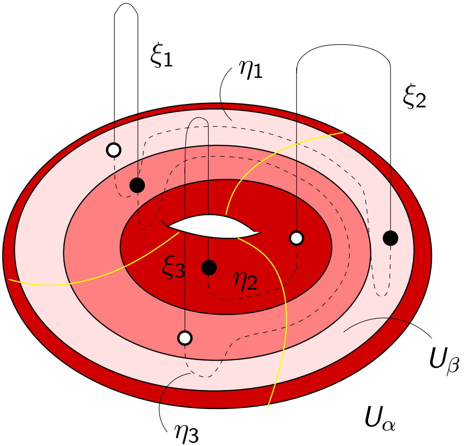
η_i is an arc in B_i from w_i to $z_{\nu(i)}$.

Push ξ_i and η_i into U_α and U_β, respectively.

$$L = \bigcup_{i=1}^{k} (\xi_i + \eta_i)$$

L is compatible with $(\Sigma_g, \alpha, \beta, \mathbf{z}, \mathbf{w})$.

Any link has a compatible multipointed Heegaard Diagram.



A 6-pointed, genus 1 Heegaard Diagram compatible with an unknot.

Chain groups and differentials

- $CFK^-(\Sigma_g, \alpha, \beta, \mathbf{w}, \mathbf{z})$ is the free module over $\mathbb{F}[U_1, .., U_k]$ generated by $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and endowed with

$$\partial^- \mathbf{x} = \sum_{\mathbf{y}} \sum_{\substack{\phi \in \pi_2(x,y): \\ \mu(\phi)=1}} \# \frac{\mathcal{M}(\phi)}{\mathbb{R}} U_1^{n_{w_1}} \cdots U_k^{n_{w_k}} \cdot \mathbf{y}$$

Where:

$n_{w_i}(\phi)$ counts the local multiplicity of ϕ at the point w_i.
$\mathcal{M}$ is the moduli space of holomorphic representatives of ϕ with dimension $\mu(\phi)$.
$\#$ counts holomorphic disks mod 2.
$\pi_2(\mathbf{x}, \mathbf{y})$ is the set of homotopy classes of Whitney disks connecting $\mathbf{x}$ to $\mathbf{y}$.

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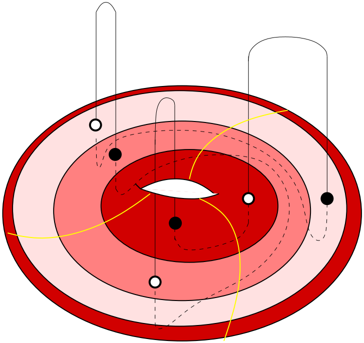
  $\#$ counts holomorphic disks mod 2.
  $\pi_2(\mathbf{x}, \mathbf{y})$ is the set of homotopy classes of Whitney disks connecting $\mathbf{x}$ to $\mathbf{y}$.

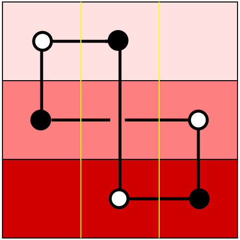
- $(C(\Sigma_g, \alpha, \beta, \mathbf{w}, \mathbf{z}), \partial)$ is the quotient complex $CFK^{-}(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})/\{U_i = 0\})$, endowed with

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}): \\ \mu(\phi)=1, n_{w_i}(\phi)=0=n_{z_i}(\phi)\ \forall i}} \# \frac{\mathcal{M}(\phi)}{\mathbb{R}} \cdot \mathbf{y}$$

Chain groups (Steps 1 - 2)

Γ is a special case of $(\Sigma_g, \alpha, \beta, \mathbf{w}, \mathbf{z})$ with $g = 1$.





Alexander grading (Step 3)

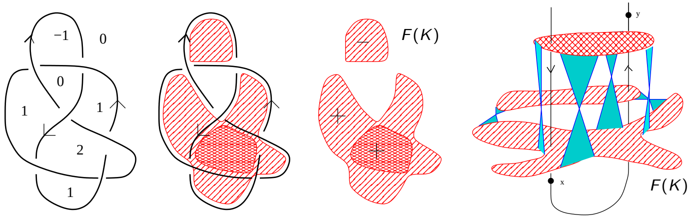
Definition: Relative Alexander grading for $CFK^-(\Sigma_g, \alpha, \beta, \mathbf{w}, \mathbf{z})$:

$$A(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{k} n_{z_i}(\phi) - \sum_{i=1}^{k} n_{w_i}(\phi)$$

We want to show A agrees with $\mathcal{A}$.

$$\begin{aligned} A(\mathbf{x}, \mathbf{y}) &= \#(K \cap D(\phi)) \\ &= \ell k(K, \gamma_{\mathbf{x},\mathbf{y}} = \partial D(\phi)) \\ &= \#(F(K) \cap \gamma_{\mathbf{x},\mathbf{y}}) \end{aligned}$$

Deform $\gamma_{\mathbf{x},\mathbf{y}}$ with horizontals under $\mathcal{T}$, verticals above $\mathcal{T}$.



$$
\begin{aligned}
&= \#(F(K) \cap \gamma_{\mathbf{x},\mathbf{y}}) \\
&= \sum \text{pos. int.} - \sum \text{neg. int.} \\
&= \sum_{y_i \in \mathbf{y}} \omega(y_i) - \sum_{x_i \in \mathbf{x}} \omega(x_i) \\
&= \sum_{x_i \in \mathbf{x}} a(x_i) - \sum_{y_i \in \mathbf{y}} a(y_i) \\
&= \mathcal{A}(\mathbf{x}) - \mathcal{A}(\mathbf{y})
\end{aligned}
$$

Maslov grading (Step 4)

Definition: Relative Maslov grading for $CFK^{-}(\Sigma_g, \alpha, \beta, \mathbf{w}, \mathbf{z})$:

$$M(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2 \sum_{i=1}^{k} n_{w_i}(\phi)$$

We want to show $\mathcal{M}$ agrees with M. This will follow if

$$\mu(\phi) = P_{\mathbf{x}}(D(\phi)) + P_{\mathbf{y}}(D(\phi))$$

where $D(\phi)$ is the formal two-chain associated with $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$.

Two facts of complex analysis

Suppose that the domain associated with ϕ is a rectangle.

1. The dimension of the moduli space of complex structures on a disk with four marked boundary points is one dimensional. (This is a special case of Lipshitz's formula for Maslov index).
2. Given a complex structure on $Sym^k(\mathcal{T})$, the moduli space of pseudo-holomorphic representatives of ϕ consists of a single representative. The unparameterized moduli space $\frac{\mathcal{M}(\phi)}{\mathbb{R}}$ corresponds with involutions of r. There is one unique involution.

These facts tell us that if $D(\phi)$ is a rectangle, $\mu(\phi) = 1$ and $\#\frac{\mathcal{M}(\phi)}{\mathbb{R}} = 1$.

Suppose $D(\phi) = r$. Then

$$\mu(\phi) = 1 = P_{\mathbf{x}}(D(\phi)) + P_{\mathbf{y}}(D(\phi))$$

Given any $\mathbf{x}$ and $\mathbf{y}$, construct a sequence $\mathbf{x} = \mathbf{x}_1, \cdots, \mathbf{x}_m = \mathbf{y}$, with corresponding Whitney disks $\phi_i \in \pi_2(\mathbf{x}_i, \mathbf{x}_{i+1})$, each of which has $D(\phi_i) = r$. Denote this concatenation of rectangles by

$$\psi = \phi_1 * \cdots * \phi_m$$

$\psi \in \pi_2(\mathbf{x}, \mathbf{y})$.

Since $\mu(\phi)$ is additive under the concatenation of Whitney disks, we have

$$\begin{aligned}\mu(\psi) &= \mu(\phi_1 * \cdots * \phi_m)\\ &= \mu(\phi_1) + \cdots + \mu(\phi_m)\\ &= \underbrace{P_{\mathbf{x}}(D(\phi_1)) + P_{\mathbf{y}}(D(\phi_1)) + \cdots + P_{\mathbf{x}}(D(\phi_m)) + P_{\mathbf{y}}(D(\phi_m))}_{\text{we want additivity here}}\end{aligned}$$

Given additivity and the fact that each $\mu(r) = 1$, we will achieve

$$\begin{aligned}\mu(\psi) &= P_{\mathbf{x}}(D(\phi_1 * \cdots * \phi_m)) + P_{\mathbf{y}}(D(\phi_1 * \cdots * \phi_m))\\ &= P_{\mathbf{x}}(D(\psi)) + P_{\mathbf{y}}(D(\psi))\end{aligned}$$

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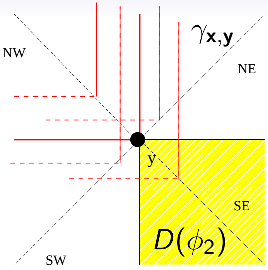
Now given $\phi_1 \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\phi_2 \in \pi_2(\mathbf{y}, \mathbf{z})$ then

$$P_x(D(\phi_1)) + P_y(D(\phi_1)) + P_y(D(\phi_2)) + P_z(D(\phi_2)) = P_x(D(\phi_1 * \phi_2)) + P_z(D(\phi_1 * \phi_2))$$

if and only if

$$\sum_{y \in \mathbf{y}} p_y(D(\phi_1)) - \sum_{z \in \mathbf{z}} p_z(D(\phi_1)) = \sum_{x \in \mathbf{x}} p_x(D(\phi_2)) - \sum_{y \in \mathbf{y}} p_y(D(\phi_2))$$



Thus

$$\begin{aligned}
& \textstyle\sum_{y\in\mathbf{y}} p_y(D(\phi_1)) - \sum_{z\in\mathbf{z}} p_z(D(\phi_1)) && = \sum_{x\in\mathbf{x}} p_x(D(\phi_2)) - \sum_{y\in\mathbf{y}} p_y(D(\phi_2)) \\
\Leftrightarrow \quad & \text{average } \#(\gamma^*_{\mathbf{y},\mathbf{z}} \cap D(\phi_1)) && = \text{average } \#(\gamma^*_{\mathbf{x},\mathbf{y}} \cap D(\phi_2)) \\
\Leftrightarrow \quad & \text{average } \ell k(\gamma^*_{\mathbf{y},\mathbf{z}}, \gamma_{\mathbf{x},\mathbf{y}}) && = \text{average } \ell k(\gamma^*_{\mathbf{x},\mathbf{y}}, \gamma_{\mathbf{y},\mathbf{z}})
\end{aligned}$$

- The symmetry of linking number verifies additivity.
- Moreover, since ϕ and ψ differ by copies of the annuli A_i and B_i, it follows that $\mathcal{M}(\phi) = \mathcal{M}(\psi)$.

Differentials (Step 5)

To identify

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}): \\ \mu(\phi)=1, n_{w_i}(\phi)=0=n_{z_i}(\phi)\ \forall i}} \# \frac{\mathcal{M}(\phi)}{\mathbb{R}} \cdot \mathbf{y}$$

with ∂ for $C(\Gamma)$, we must verify that:

1. The only domains $D(\phi)$ satisfying

$$\phi \in \pi_2(\mathbf{x},\mathbf{y}) \mid \mu(\phi) = 1, n_{w_i}(\phi) = 0 = n_{z_i}(\phi)\ \forall i$$

   are rectangles.

2. $\# \frac{\mathcal{M}(\phi)}{\mathbb{R}} = 1$ or 0.

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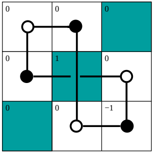
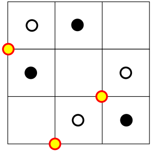
$$\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi) = 1, n_{w_i}(\phi) = 0 = n_{z_i}(\phi)\ \forall i$$

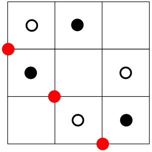
   are rectangles.
   - Results from additivity and the requirement $n_p(\phi) \geq 0$.
2. $\# \frac{\mathcal{M}(\phi)}{\mathbb{R}} = 1$ or 0.
   - Results from Fact 2.

Example: the unknot.

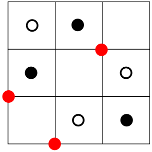




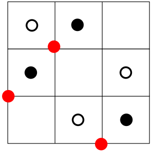
$M(x_0) = -2$
$A(x_0) = -2$



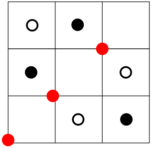
$M(x_1) = -1$
$A(x_1) = -1$



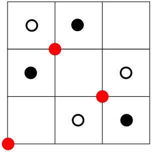
$M(x_2) = -1$
$A(x_2) = -1$



$M(x_3) = 0$
$A(x_3) = 0$



$M(x_4) = 0$
$A(x_4) = -1$



$M(x_5) = -1$
$A(x_5) = -1$

- r_1 connects x_4 to x_2.
- r_2 connects x_4 to x_5.
- r_3 connects x_4 to x_1.

$$0 \rightarrow < \underset{0}{x_3, x_4} > \xrightarrow{\partial} < \underset{-1}{x_1, x_2, x_5} > \xrightarrow{\partial' = 0} < \underset{-2}{x_0} > \rightarrow 0$$

$$\partial x_3 = 0, \partial x_4 = x_2 + x_5 + x_1 \quad \Rightarrow \quad Ker(\partial) \cong \mathbb{Z}_2 \text{ and } Im(\partial) \cong \mathbb{Z}_2$$

$$\partial' x_1 = \partial' x_2 = \partial' x_5 = 0 \quad \Rightarrow \quad Ker(\partial') \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \text{ and } Im(\partial) = 0$$

Thus

$$H_0(\Gamma_U) \cong \mathbb{Z}_2, \; H_{-1}(\Gamma_U) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \; H_{-2}(\Gamma_U) \cong \mathbb{Z}_2$$

Verifying the theorem, we use the fact that

$$\widehat{HFK}(\Gamma_U) \cong \mathbb{Z}_2$$

Then with $V = \mathbb{Z}_{2_{(0,0)}} \oplus \mathbb{Z}_{2_{(-1,-1)}}$, and with $n = 3$ we obtain

$$\begin{aligned}
\widehat{HFK}(S^3; \Gamma_U) \otimes V \otimes V &= \mathbb{Z}_2 \otimes_{\mathbb{Z}_2} [\mathbb{Z}_{2_{(0,0)}} \oplus \mathbb{Z}_{2_{(-1,-1)}}] \\
&\quad \otimes_{\mathbb{Z}_2} [\mathbb{Z}_{2_{(0,0)}} \oplus \mathbb{Z}_{2_{(-1,-1)}}] \\
&= [\mathbb{Z}_{2_{(0,0)}} \oplus \mathbb{Z}_{2_{(-1,-1)}}] \otimes_{\mathbb{Z}_2} [\mathbb{Z}_{2_{(0,0)}} \oplus \mathbb{Z}_{2_{(-1,-1)}}] \\
&= \mathbb{Z}_{2_{(0,0)}} \oplus \mathbb{Z}_{2_{(-1,-1)}} \oplus \mathbb{Z}_{2_{(-1,-1)}} \oplus \mathbb{Z}_{2_{(-2,-2)}}
\end{aligned}$$

which is consistent with our computation.

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* Extra Slides *

Absolute Alexander grading

The graded Euler characterisic of $\widehat{HFK}$ is the Alexander polyonial:

$$\Delta_K(T) = \sum_{s \in \mathbb{Z}} \chi(\widehat{HFK}_*(K, s))) \cdot T^s$$

We use this known formula to pin down the additive indeterminacy of $A(\mathbf{x})$, by requiring that $A : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}$ be defined so that the graded Euler characteristic is the *symmetrized* Alexander polynomial. In fact, we require

$$\widehat{HFK}_*(K, s) \cong \widehat{HFK}_*(K, -s)$$

Absolute Maslov grading

If we ignore the k basepoints $\mathbf{z}$ we obtain a k-pointed Heegaard diagram $(\Sigma_g, \alpha, \beta, \mathbf{w})$. The corresponding chain complex CF^- has a quotient complex, $CF^-/\{U_i = 0\}$, for which

$$H_*(CF^-/\{U_i = 0\}) = \widehat{HF}(Y\#(\#^{k-1}S^2 \times S^1))$$

We can pin down the Maslov grading (for our purposes) to ensure this isomorphism. Note that

$$M(\mathbf{x}) - M(\mathbf{y}) = \mu(\phi) - 2\sum n_{w_i}(\phi)$$

does not depend on the knot or basepoints $\mathbf{z}$.