Computations with Heegaard Floer Homology

Allison Moore

November 19, 2008

What we're talking about today

Y is an oriented, closed 3-manifold. We will associate to Y a finitely generated abelian group: $Y \rightsquigarrow HF(Y)$.

- Today: the computation of this three manifold invariant (**Oszvath and Szabo**).

What we're talking about today

Y is an oriented, closed 3-manifold. We will associate to Y a finitely generated abelian group: $Y \rightsquigarrow HF(Y)$.

- Today: the computation of this three manifold invariant (**Oszvath and Szabo**).
- Future: Knot Heegaard Floer homology induces a filtration on homology groups, and the filtered homotopy type categorifies the Alexander polynomial (recall that Khovanov homology categorifies the Jones polynomial)

The purpose of this talk is

- To provide an introduction to Heegaard Floer homology.
- To compute "simple example".
- To convince you to take up Heegaard Floer homogy in your research interests.
- To learn how to make slides.

Heegaard decompositions

Heegaard decomposition (Σ_g, U_0, U_q): write $Y = U_0 \cup_{\Sigma_g} U_1$.

For example:

- The genus 0 decomposition of S^3 into two balls.
- The genus 1 decomposition of S^3 into two solid tori.
- Lens spaces (genus 1).

Heegaard decompositions

Heegaard decomposition (Σ_g, U_0, U_q): write $Y = U_0 \cup_{\Sigma_g} U_1$.

For example:

- The genus 0 decomposition of S^3 into two balls.
- The genus 1 decomposition of S^3 into two solid tori.
- Lens spaces (genus 1).

Important facts:

- Y has a Heegard decomposition.
- Any two Heegaard decompositions of Y are connected by stabilizations.

These facts imply invariants of Heegaard decompositions are invariants of three manifolds.

Heegaard diagrams

- Definition: Heegard diagram:
  Start with a Heegaard decomposition.
  A Heegaard diagram is Σ_g together with a collection of attaching circles
  $$\alpha = (\alpha_1, \cdots, \alpha_g)$$
  $$\beta = (\beta_1, \cdots, \beta_g)$$
  for U_0 and U_1, respectively.
- **Example:** Check out the board.

Morse theory plays an important role in Heegaard Floer homology

- Y admits a self indexing Morse function.
- **Prop**: Given a self indexing Morse function $f : Y \longrightarrow [0, 3]$ we can produce a Heegaard diagram from ∇f.
- 4 manifold cobordism theory makes Knot Heegard Floer homology into a bigraded abelian group.

A configuration space

To define the upcoming boundary maps, we will make use of a space related to Σ_g.

- $Sym^g(\Sigma_g)$ is the space of unordered g-tuples of points in Σ_g.
- $\alpha_1 \times \cdots \times \alpha_g$ is the g-torus $\mathbb{T}_\alpha$ inside $Sym^g(\Sigma_g)$.
  $\beta_1 \times \cdots \times \beta_g$ is the g-torus $\mathbb{T}_\beta$ inside $Sym^g(\Sigma_g)$.
- **Important for us**: multipoints $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and basepoints $z \in \Sigma_g - \alpha - \beta$.

Whitney disks

Definition: We say **x** is connected to **y** by a Whitney disk if there exists a continuous map

$$u : \mathbb{D} \to Sym^g(\Sigma_g)$$

where $u(-i) = \mathbf{x}$, $u(i) = y$, $u(e_1) = \mathbb{T}_\alpha$ and $u(e_2) = \mathbb{T}_\beta$.
$\pi_2(\mathbf{x}, \mathbf{y})$ is the set of homotopy classes of such maps.

- The existence and frequency of such a map will determine how the boundary operator works.

A tool for understanding Whitney disks and multipoints

Let a and b be paths from $\mathbf{x}$ to $\mathbf{y}$ along α and β. Consider the isomorphisms:

$$\frac{H_1(Sym^g(\Sigma_g))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \cong \frac{H_1(\Sigma_g)}{\{[\alpha_i], [\beta_i]\}} \cong H_1(Y;\mathbb{Z})$$

These induce a well defined map

$$\epsilon(\mathbf{x}, \mathbf{y}) = [a - b] \in H_1(Y, \mathbb{Z})$$

- ϵ can be calculated in Σ_g (easier).

A tool for understanding Whitney disks and multipoints

Let a and b be paths from $\mathbf{x}$ to $\mathbf{y}$ along α and β. Consider the isomorphisms:

$$\frac{H_1(Sym^g(\Sigma_g))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \cong \frac{H_1(\Sigma_g)}{\{[\alpha_i],[\beta_i]\}} \cong H_1(Y;\mathbb{Z})$$

These induce a well defined map

$$\epsilon(\mathbf{x},\mathbf{y}) = [a-b] \in H_1(Y,\mathbb{Z})$$

- ϵ can be calculated in Σ_g (easier).
- $\epsilon(\mathbf{x},\mathbf{y})$ is nontrivial $\Rightarrow \pi_2(\mathbf{x},\mathbf{y})$ is empty, meaning no Whitney disks connect $\mathbf{x}$ and $\mathbf{y}$.

A tool for understanding Whitney disks and multipoints

Let a and b be paths from $\mathbf{x}$ to $\mathbf{y}$ along α and β. Consider the isomorphisms:

$$\frac{H_1(Sym^g(\Sigma_g))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \cong \frac{H_1(\Sigma_g)}{\{[\alpha_i], [\beta_i]\}} \cong H_1(Y; \mathbb{Z})$$

These induce a well defined map

$$\epsilon(\mathbf{x}, \mathbf{y}) = [a - b] \in H_1(Y, \mathbb{Z})$$

- ϵ can be calculated in Σ_g (easier).
- $\epsilon(\mathbf{x}, \mathbf{y})$ is nontrivial $\Rightarrow \pi_2(\mathbf{x}, \mathbf{y})$ is empty, meaning no Whitney disks connect $\mathbf{x}$ and $\mathbf{y}$.
- $\mathbf{x} \sim \mathbf{y}$ if $\epsilon(\mathbf{x}, \mathbf{y}) = 0$.

Maslov index

If ϕ is a homotopy class in $\pi_2(\mathbf{x}, \mathbf{y})$, let $\mathcal{M}(\phi)$ denote the moduli space of holomorphic representatives of ϕ. Let $\widehat{\mathcal{M}}(\phi) = \frac{\mathcal{M}(\phi)}{\mathbb{R}}$ denote the unparameterized version of $\mathcal{M}(\phi)$. $\mu(\phi)$ refers to the dimension of $\mathcal{M}(\phi)$.

- We will stick to the case where $\mu(\phi) = 1$, so that $\widehat{\mathcal{M}}(\phi)$ is a signed collection of points.
- μ is independent of choice of ϕ.
- The good news is that given a Whitney disk, there is a combintorial formula for its Maslov index.
- The bad news is that I don't know the formula (yet).

$Spin^c$- structures

- Two nonvanishing vector fields are homologous if they are homotopic outside of a ball.
- $Spin^c(Y)$ is the space of nonvanishing vector fields over Y modulo this relation.
- **Prop**: There is a 1-1 correspondence $H^2(Y,\mathbb{Z}) \longleftrightarrow Spin^c(Y)$.

How we use $Spin^c$

- We want to assign a $Spin^c$ structure to each multipoint:

$$s_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rightarrow Spin^c(Y)$$

This map is described by ∇f, much like the Heegaard diagram.

How we use $Spin^c$

- We want to assign a $Spin^c$ structure to each multipoint:

$$s_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rightarrow Spin^c(Y)$$

  This map is described by ∇f, much like the Heegaard diagram.

- **Prop:** Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Then

$$s_z(\mathbf{y}) - s_z(\mathbf{x}) = \text{Poincare dual } [\epsilon(\mathbf{x}, \mathbf{y})]$$

Several homology variants

Although we will be focusing on $\widehat{HF}$ and HF^{∞} there are many similar variants

- $\widehat{HF}$
- HF^{+}
- HF^{-}
- HF^{∞}
- HF_{red}

For all of these homologies we will need a "pointed" Heegaard diagram

$$(\Sigma, \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g, z)$$

We will also need to settle on a single $Spin^c$ structure s.

Chain groups

$\widehat{CF}(\alpha, \beta, s)$ is the free abelian group generated by multipoints $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $s_z(\mathbf{x}) = s$.
Relative grading for $\widehat{CF}$:

$$gr(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2n_z(\phi)$$

Chain groups

$\widehat{CF}(\alpha, \beta, s)$ is the free abelian group generated by multipoints $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $s_z(\mathbf{x}) = s$.
Relative grading for $\widehat{CF}$:

$$gr(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2n_z(\phi)$$

$CF^\infty(\alpha, \beta, s)$ is the free abelian group generated by pairs $[\mathbf{x}, i]$ with $s_z(\mathbf{x}) = s, i \in \mathbb{Z}$.
Relative grading for CF^∞:

$$gr([\mathbf{x},, i], [\mathbf{y}, j]) = gr(\mathbf{x}, \mathbf{y}) + 2i - 2j$$

Boundary maps

The boundary maps are similar

For $\widehat{CF}$:

$$\hat{\partial}\mathbf{x} = \sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta \\ :s_z(\mathbf{y})=s}} \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y}) \\ :n_z(\phi)=0}} c(\phi)\cdot\mathbf{y}$$

$n_z(\phi) = \#\phi^{-1}(\{z\}\times Sym^g(\Sigma_g))$.
$c(\phi) =$ number of points in $\widehat{\mathcal{M}}(\phi)$ if $\mu(\phi) = 1$, $c(\phi) = 0$ otherwise.

Boundary maps

The boundary maps are similar

For $\widehat{CF}$:

$$\hat{\partial}\mathbf{x} = \sum_{\substack{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta \\ :s_z(\mathbf{y})=s}} \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y}) \\ :n_z(\phi)=0}} c(\phi)\cdot\mathbf{y}$$

For CF^∞:

$$\partial^\infty[\mathbf{x}, i] = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\phi\in\pi_2(\mathbf{x},\mathbf{y})} c(\phi)\cdot[\mathbf{y}, i - n_z(\phi)]$$

$n_z(\phi) = \#\phi^{-1}(\{z\}\times Sym^g(\Sigma_g))$.
$c(\phi) =$ number of points in $\widehat{\mathcal{M}}(\phi)$ if $\mu(\phi) = 1$, $c(\phi) = 0$ otherwise.

Surgeries on the trefoil

$Y = Y_n$, the three manifold obtained from surgery on the (2,3) torus knot.

- Assume $n > 6$ and $n = k - 6$.
- Let z_5 be our basepoint.

Attach along β_1 to obtain the complement of the trefoil in S^3.

$$H_1(Y, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$$

There will be n equivalence classes of intersection points. ($Spin^c \cong \mathbb{Z}/n\mathbb{Z}$).

Lemma: *The points $a = \{x_1, w_9\}$, $b = \{x_2, w_8\}$, and $c = \{x_3, w_7\}$ make up an equivalence class.*

Take a standard basis for $H_1(\Sigma_g)$: $\alpha_1, \alpha_2, B_1, B_2$.

$$[\beta_1] = -B_1 - B_2 \qquad \text{and} \qquad [\beta_2] = 2B_1 + (n+2)B_2$$

- $H_1(Y_n) \cong < h >$, where $B_1 = -B_2 = h$.
- Use this to compute $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(Y)$ for all the intersection points.
- Organize intersection points into equivalence classes.

Lemma:

There exists a $\phi \in \pi_2(c, b)$ and a $\psi \in \pi_2(a, b)$ with $\mu(\phi) = \mu(\psi) = 1$. Moreover,

$$\#\widehat{\mathcal{M}}(\phi) = \#\widehat{\mathcal{M}}(\psi) = \pm 1$$

Furthermore,

$$\begin{aligned} n_{z_r}(\phi) &= 0 && \text{for } r < i - 2 \\ n_{z_r}(\phi) &= 1 && \text{for } r \geq i - 2 \\ n_{z_r}(\psi) &= 1 && \text{for } r \leq i - 2 \\ n_{z_r}(\psi) &= 1 && \text{for } r > i - 2 \end{aligned}$$

Compute gradings

Here, $i = 7$ and $r = 5$. In terms of $\widehat{CF}$:

$$gr(a,b) = \mu(\psi) - 2n_{z_5}(\psi) = 1 - 2(1) = -1$$

$$gr(c,b) = \mu(\phi) - 2n_{z_5}(\phi) = 1 - 2(1) = -1$$

In terms of CF^∞

$$gr([a,i],[b,i-1]) = gr(a,b) + 2i - 2(i-1) = -1 + 2 = 1$$

$$gr([c,i],[b,i-1]) = gr(c,b) + 2i - 2(i-1) = -1 + 2 = 1$$

Note $U[x,i] = [x,i-1]$ decreases the relative grading by 2.

$\widehat{HF}(Y)$

$\widehat{CF}(Y)$ is described by

$$0 \rightarrow < b > \xrightarrow{\widehat{\partial}} < a, c > \rightarrow 0$$

Claim: $\widehat{\partial} = 0$. Then

$$HF_{m-1}(\widehat{HF}(Y) = < a, c > \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$HF_m(\widehat{HF}(Y) = < b > \cong \mathbb{Z}$$

$HF^\infty(Y)$

CF^∞ is described by

$$\cdots \to < [a, i+1], [c, i+1] > \xrightarrow{\partial^\infty} < [b, i] > \xrightarrow{0} < [a, i], [c, i] >$$

$$\cdots \xrightarrow{\partial^\infty} < [b, i-1] > \xrightarrow{0} < [a, i-1], [c, i-1] > \to \cdots$$

- $[b, i] \to 0$ here and above.
- $\partial^\infty([a, i]) = [b, i-1] = \partial^\infty([c, i])$

$HF^\infty(Y)$

CF^∞ is described by

$$\cdots \to < [a, i+1], [c, i+1] > \xrightarrow{\partial^\infty} < [b, i] > \xrightarrow{0} < [a, i], [c, i] >$$

$$\cdots \xrightarrow{\partial^\infty} < [b, i-1] > \xrightarrow{0} < [a, i-1], [c, i-1] > \to \cdots$$

- $[b, i] \to 0$ here and above.
- $\partial^\infty([a, i]) = [b, i-1] = \partial^\infty([c, i])$
- HF^∞ alternates between $< [a, i] - [c, i] > /0 \cong \mathbb{Z}$ and $< [b, i] > / < [b, i] >= 0$. In particular

$$HF^\infty = \mathbb{Z}[U, U^{-1}]$$

Short exact sequences

Recall that the boundary operator drops the index.

- $CF^{-}(\alpha,\beta,s) \subset CF^{\infty}(\alpha,\beta,s)$ is the subgroup generated by pairs $[x,i]$ with $i<0$.
- $CF^{+}(\alpha,\beta,s)$ is the quotient group

$$CF^{\infty}(\alpha,\beta,s)/CF^{-}(\alpha,\beta,s)$$

There exists a short exact sequence

$$0 \to CF^{-}(\alpha,\beta,s) \xrightarrow{\iota} CF^{\infty}(\alpha,\beta,s) \xrightarrow{\pi} CF^{+}(\alpha,\beta,s) \to 0$$

Related topics

- Defining a Knot Heegaard Floer homology to study knots.
- Using the bigraded abelian group to study knots.

$$\Sigma_i \Sigma_j (-1)^j \cdot \mathrm{rk}(H_{i,j}(K)) \cdot T^i = \Delta_K(T)$$

- TQFT
- Spectral sequences
- Relations with Khovanov Homology

Some facts about $Sym^g(\Sigma_g)$:

- $\pi_1(Sym^g(\Sigma)) \cong H_1(\Sigma)$ and $\pi_2(Sym^g(\Sigma)) \cong \mathbb{Z}$, $g > 2$.
- $\{z\} \times Sym^{g-1}(\Sigma_g)$ is a codim 1 subspace
- $H_1(Sym^g(\Sigma_g)) \setminus H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta) \cong H_1(Y;\mathbb{Z})$.