USE OF LITTLEWOOD-PALEY OPERATORS FOR THE EQUATIONS OF FLUID MOTION

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THESIS

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To my parents Anica, Josif and Rade,

and my husband Miloš.

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SUMMARY

In this thesis, we study a partial regularity result for a modified version of the Navier-Stokes equations. In the standard Navier-Stokes equations the Laplacian of the velocity field appears to the power one, which reflects amount of energy dissipation. The deepest regularity result for the Navier-Stokes equations to date is the result of Caffarelli, Kohn and Nirenberg which states that the Hausdorff dimension of the singular set is at most one. We consider a modification of the Navier-Stokes equations in order to investigate how system reacts to changes caused by different amount of energy dissipation. More precisely, we study the Navier-Stokes equations with hyper-dissipation, where the Laplacian is raised to a power. We get an estimate for the dimension of the set of singular points at the first time of blow-up depending on the degree of dissipation.

In chapter 1 we describe the equations of fluid motion and some of unsolved questions concerning them. Also in chapter 1 we state our main result about the Navier-Stokes equations with hyper-dissipation. In chapter 2 we review Littlewood-Paley theory which will be used in the rest of the thesis. In such a way we describe techniques needed for localization in frequency. In chapter 3 we use Littlewood-Paley operators to present some classical results of fluid dynamics such as (1), which gives a criterion for loss of regularity for the solutions of the Euler equations in 3D. In chapter 4 we introduce a dyadic model for the equations of fluid motion. The dyadic model possesses an important feature of the equations, which is conservation (or decay) of energy. For such a model we prove partial regularity results for the Navier-Stokes equations with hyper-dissipation and also for the Ladyzhenskaya's modification of the Navier-Stokes equations, in which case the dissipation term is nonlinear.

The goal of chapters 5, 6 and 7 is to generalize the ideas of partial regularity result from the dyadic Navier-Stokes equations with hyper-dissipation presented in chapter 4 to the actual Navier-Stokes equations with hyper-dissipation. We do that by combining techniques of Littlewood-Paley operators and the theory of pseudodifferential operators. In chapter 5 we describe pseudodifferential operators which help us to localize in space, and we develop calculus needed to localize our equations in both frequency and space. We define a bad set in this chapter and show how one can cover such a bad set and estimate its Hausdorff dimension. In chapter 6 we figure out a balance between the nonlinear term and the dissipation term for a localized equation, and outside of the bad set we prove a certain level of regularity called critical regularity. Then in chapter 7 we finish the proof of partial regularity result for the Navier-Stokes equations with hyper-dissipation by proving an arbitrary level of regularity inside a cube for which one has a little bit better regularity then critical one.

In chapter 8 we revisit dyadic models introduced in chapter 4, and for the dyadic Euler equations we prove finite time blow-up, which we prove for the Navier-Stokes equations with sufficiently small degree of dissipation too. However these results are built on a lower bound for the nonlinear term, and are valid only for the dyadic models.

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CHAPTER 1

INTRODUCTION

1.1 The Euler and Navier-Stokes equations

The partial differential equations that describe the most crucial properties of the fluid motion are the Euler equations. They are derived for an ideal fluid, by which one means:

- incompressible (fluid whose particle do not change volume as the fluid moves)
- inviscid (fluid without internal friction between particles)
- fluid with constant density.

Let $x = (x_1, x_2, ..., x_n)$ denote a point in $\mathbb{R}^n$. Let the vector $u = u(x,t) = (u_1(x,t), ..., u_n(x,t)) \in \mathbb{R}^n$ be the velocity field, and $p = p(x,t) \in \mathbb{R}$ the pressure. Then the Euler equations are:

$$\frac{Du}{Dt} = -\nabla p, \tag{1.1.1}$$

$$\nabla \cdot u = 0, \tag{1.1.2}$$

with the initial condition

$$u(x, 0) = u_0(x). \tag{1.1.3}$$

Here we used the notation $\frac{D}{Dt}$ which is defined for some fluid quantity $f = f(x,t)$ by

$$\frac{Df}{Dt} := \frac{d}{dt} f(x_1(t), x_2(t), ...x_n(t), t),$$

and it represents a rate of change of f. The symbol $\frac{D}{Dt}$ is often called "material derivative" or "convective derivative".

By chain rule we have

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (u \cdot \nabla) f,$$

and in particular

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla) u.$$

Notice that the equation (1.1.1) expresses Newton's second law $F = ma$. Indeed the right-hand side of (1.1.1) represents the force which is in this case the force of internal pressure, while the left-hand side of (1.1.1) is the acceleration. The mass is 1, since the density is constant and taken to be 1. On the other hand, the equation (1.1.2) reveals conservation of mass.

When the Euler equations are considered in $\mathbb{R}^n$ one would like to avoid the situation in which the velocity field $u(x,t)$ becomes unbounded as $|x| \to \infty$. Usually one restricts behavior of the velocity field at infinity by imposing the bounded energy condition:

$$\int_{\mathbb{R}^n} |u(x,t)|^2 dx < constant, \tag{1.1.4}$$

for all time $t \geq 0$. This is a natural condition to assume, because from the first look at the equation (1.1.1) we see neither obvious dissipative effects nor an effect of an external force. In this thesis whenever we consider either the Euler equations or a model for the Euler equations we shall assume the bounded energy (1.1.4).

Despite the fact that Euler introduced the equations (1.1.1) - (1.1.3) in 1755, some basic questions concerning them are still unsolved. For example, it is an outstanding problem of fluid dynamics to find out if solutions of the Euler equations satisfying (1.1.4) form singularities in finite time. The answer is "no" in the case of Euler equations in dimension two. But in three-dimensional space the question is still open. However the local existence theorem for Euler's equations is known (27), (28) as well as (15), (16), (17), and we will state a version of the theorem in Chapter 2 where we discuss some classical results about Euler equations. For details on local existence of smooth solutions of the Euler equations see, for example, (19) or (48).

The equations that describe the most fundamental properties of viscous fluids are the Navier-Stokes equations which are given with the following equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nu \Delta u + f, \tag{1.1.5}$$

$$\nabla \cdot u = 0, \tag{1.1.6}$$

and the initial condition

$$u(x,0) = u_0(x). \tag{1.1.7}$$

When the Navier-Stokes equations are considered in $\mathbb{R}^n$ one imposes the restriction that the energy is bounded (1.1.4).

As with the Euler equations the theory of the Navier-Stokes equations in three dimensions is far from being complete. The major open problem is the question of global existence of smooth solutions of the Navier-Stokes equations satisfying (1.1.4) in 3D, where by a smooth solution in $\mathbb{R}^n$ one means a solution to the Navier-Stokes equations such that $u(x,t) \in C^\infty[\mathbb{R}^n \times \mathbb{R}^+]$. For the precise formulation of this open problem see (12).

The existence of smooth solutions of the Navier-Stokes equations initial boundary value problem in 3D has been proved locally in time, see, for example, (19), (22), (11), (13), (21), (46), (6). Also global existence of smooth solutions of the Navier-Stokes equations in 3D has been proved provided small initial data. This was addressed by various authors in different function spaces, see, for example, (20), (13), (46), (5), (35), (23).

In 1930s Leray (24) - (26) introduced a notion of weak solutions of the Navier-Stokes equations. A weak solution of the Navier-Stokes equations is introduced as a function u which satisfies the following equation:

$$\langle \frac{\partial u}{\partial t}, \phi \rangle + \langle u \cdot \nabla u, \phi \rangle + \langle \nabla p, \phi \rangle = \nu \langle \Delta u, \phi \rangle + \langle f, \phi \rangle, \tag{1.1.8}$$

for all smooth functions ϕ compactly supported in $\mathbb{R}^3 \times (0, \infty)$, where $\langle \cdot, \cdot \rangle$ stands for the scalar product on $L^2(\mathbb{R}^3 \times \mathbb{R})$. Obviously if functions u and p satisfy the Navier-Stokes equations (1.1.5) - (1.1.6) in a strong sense then they satisfy (1.1.8). On the other hand, an advantage of having (1.1.8) is that by using integration by parts on (1.1.8), one can move derivatives to fall on the test function ϕ.

Leray (26) and Hopf (18) showed existence of a weak solution of the Navier-Stokes equations satisfying the following energy inequality:

$$\int_{\Omega \times \{t\}} |u|^2 + 2\int_0^t \int_\Omega |\nabla u|^2 dx dt \leq \int_\Omega |u(x,0)|^2 dx + 2\int_0^t \int_\Omega f \cdot u, \tag{1.1.9}$$

where Ω is a bounded domain in $\mathbb{R}^3$. The inequality (1.1.9) is obtained by pairing in $L^2(\mathbb{R}^3 \times \mathbb{R})$ the equation (1.1.5) with $2u\phi$, then integrating by parts and taking $\phi \equiv 1$.

Regularity of weak solutions was investigated by many authors, for example, Serrin (41), (42), Scheffer (36) - (40), Caffarelli-Kohn-Nirenberg (4), Lin (29). We shall discuss a particular type of regularity results known as partial regularity results which give an upper bound on the Hausdorff dimension of the singular set for an equation, see for example (4), (29).

In (4) Caffarelli, Kohn and Nirenberg introduced a particular class of weak solutions of the Navier-Stokes equations called suitable weak solutions. By a suitable weak solution they mean a weak solution of the Navier-Stokes equations such that for each bump function ϕ compactly supported in space and time the following inequality is valid:

$$2\int\int |\nabla u|^2 \phi \leq \int\int [|u|^2(\phi_t + \Delta\phi) + (|u|^2 + 2p)u \cdot \nabla\phi + 2(u \cdot f)\phi]. \tag{1.1.10}$$

The inequality (1.1.10) is known as generalized energy inequality. It is important to notice that the first term on the right-hand side can be made small by choosing ϕ to satisfy the backwards heat equation, and this was used in (4). Caffarelli, Kohn and Nirenberg (4) proved that for the Navier-Stokes equations (1.1.5) - (1.1.7) the singular set of a suitable weak solution has parabolic Hausdorff dimension at most 1. Here parabolic Hausdorff dimension is defined in an analogous way to Hausdorff dimension, just by using parabolic cylinders Q_r instead of Euclidean balls, where

$$Q_r(x,t) = \{(y,\tau) : |y-x| < r,\ t-r^2 < \tau < t\}.$$

In order to prove a partial regularity result for the class of suitable weak solutions they first proved a local dimensionless result which states that if u, p and f are small enough on the parabolic cylinder Q_r then u is regular on the smaller cylinder $Q_{r/2}$. As a consequence of this they obtained an estimate of the minimum rate at which a singularity could develop and they proved a sufficient condition for a point to be a regular point. By covering the singular set they obtained that for any suitable weak solution, the singular set has one-dimensional parabolic Hausdorff measure zero. A new proof of this result was recently given by Lin (29). Lin also uses the generalized energy inequality (1.1.10). Our motivation came partly from (4) and we explain that in the next section.

1.2 Partial regularity results for the Navier-Stokes equations with hyper-dissipation

The Navier Stokes equation with dissipation $(-\Delta)^\alpha$ is given by

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = -(-\Delta)^\alpha u, \tag{1.2.1}$$

where u is a time-dependent divergence free vector field in $\mathbb{R}^3$. One sets the initial condition

$$u(x,0) = u_0(x) \tag{1.2.2}$$

where $u_0(x) \in C_c^\infty(\mathbb{R}^3)$.

If we pair (in $L^2(\mathbb{R}^3 \times \mathbb{R})$) the equation (1.2.1) with u and then integrate by parts, we see that classical solutions to this equation on a time interval $[0,T]$ satisfy conservation of energy, namely that

$$||u(.,T)||_{L^2}^2 = ||u_0||_{L^2}^2 - \int_0^T \langle (-\Delta)^\alpha u, u \rangle.$$

The second term on the right is called the dissipation term.

Caffarelli, Kohn, and Nirenberg (4) showed that when $\alpha = 1$, the singular set of a generalized weak solution to the system (1.2.1),(1.2.2) has parabolic Hausdorff dimension at most 1. This could be considered a first step towards showing global strong solvability. Any improvement in this upper bound on the dimension would be genuine progress towards solution of the global solvability problem.

Another well known fact is the following. We learned the proof which we present in the chapter 2 from Diego Cordoba. J.L. Lions gave a version of the proof in (30), (31). Also Mattingly and Sinai (32) recently gave a different proof.

Proposition 1.2.1 *If $\alpha \geq \frac{5}{4}$, one has global strong solvability for the system* (1.2.1),(1.2.2).

Indeed proposition 1.2.1 could be also viewed as a first step towards the solution of the global strong solvability for $\alpha = 1$ and any improvement in the exponent $\frac{5}{4}$ could be viewed as genuine progress.

We (Katz, N., and Pavlović, N.: A cheap Caffarelli-Kohn-Nirenberg inequality for the Navier-Stokes equation with hyper-dissipation. To appear in *Geometric and Functional analysis*, 2002) interpolate the two results so that these two paths to progress are unified. We prove

Theorem 1.2.2 *If T is the time of first breakdown for the system* (1.2.1),(1.2.2), *with* $1 < \alpha < \frac{5}{4}$ *then the Hausdorff dimension of the singular set at time T is at most* $5 - 4\alpha$.

Heuristically, the theorem ought to be a mild generalization of Proposition 1.2.1. We think about the microlocal analysis of a solution u in terms of coefficients u_Q associated to cubes Q. Very roughly speaking, the coefficient u_Q should be viewed as a generalized wavelet coefficient. The proof of Proposition 1.2.1 goes wrong for $\alpha < \frac{5}{4}$ only because of a small number of cubes with large coefficients. At any time, the set of points contained in arbitrarily small such cubes has dimension at most $5 - 4\alpha$.

We were also unable to directly generalize the proof of Caffarelli, Kohn, and Nirenberg. They rely on the "generalized energy inequality" which is built on an amusing property of the divergence free heat equation. Let ϕ be any compactly bump function in space and time and u a divergence free vector field, then

$$\int \langle (\frac{\partial}{\partial t} - \Delta) u, \phi u \rangle dt = \int (\frac{1}{2}\frac{\partial}{\partial t}(\langle u, \phi u \rangle) + \langle \nabla u, \phi \nabla u \rangle - \langle (\frac{1}{2}\frac{\partial \phi}{\partial t} + \Delta \phi) u, u \rangle) dt.$$

The first term represents a change in local energy. The second term represents local dissipation. The third term is an error which can be made insignificant by choosing ϕ to satisfy the backwards heat equation. It is this method of controlling the error which we are unable to generalize.

To circumvent the problem of not having the "generalized energy inequality" we utilize techniques of Littlewood-Paley theory which was developed by Bony (see e.g. (3)), and Coifman and Meyer (see e.g.(9)), and pseudodifferential operators of type $(1, 1-\epsilon)$, which are described in, for example, (47). In such a way we localize in frequency and in space. We have a "quasi" version of the "generalized energy inequality" which works for certain neighboring cubes and allows us to prove a critical level of regularity outside of "bad" cubes in which too much dissipation occurs. Then we prove a barrier estimate which guarantees arbitrary regularity in the interior of a cube Q, if the critical regularity is known for cubes containing it and for boundary cubes of the cube Q. This barrier estimate can be thought of as a localized version of global existence with small data. (One can compare this estimate with results in (23) and (5) in which local well posedness is established with hypotheses slightly more relaxed than ours.) But in order for this to work, it is important that there not be too many small dissipating cubes on the boundary of our cube. This combinatorial issue is an ingredient which seems not to have appeared before and which restricts us to the case $\alpha > 1$. On the other hand we cover the "bad" set and from the covering we can estimate an upper bound for the Hausdorff dimension of the singular set.

Remark on notation

Throughout the thesis the expression $A \lesssim B$ means $A \leq cB$ where c is a constant. Such a constant c may depend on the time of first blow-up and on the initial conditions on the velocity. It does not depend on a particular scale or on a cube where we are estimating.

CHAPTER 2

LITTLEWOOD-PALEY OPERATORS

Here we present a review of Littlewood-Paley operators by following the lecture-notes of Terence Tao (45) and the book of Stein (43). Our presentation is brief and suits the purposes of the thesis. For details see, for example, elegant expositions in (10), (43), (45).

In section 2.1 we introduce Littlewood-Paley operators and prove Bernstein's and the "cheap Littlewood-Paley" inequalities. Then in section 2.2 by using Littlewood-Paley operators we present a proof of a version of the Sobolev embedding theorem in $\mathbb{R}^3$. In section 2.3 we show how one can decompose a product of two functions by looking at pieces localized in frequencies.

2.1 Introduction to Littlewood-Paley operators

We shall use a standard Littlewood-Paley partition of frequency space in $\mathbb{R}^3$. Let σ_0 be a smooth bump function supported on $|\xi| \leq 2$, and such that $\sigma_0(\xi) = 1$ for $|\xi| \leq 1$. Then we define the function $p_0(\xi)$ by

$$p_0(\xi) = \sigma_0(\xi) - \sigma_0(2\xi).$$

Therefore the function $p_0(\xi)$ is smooth and supported on $\frac{1}{2} \leq |\xi| \leq 2$.

Let us introduce functions $p_j(\xi)$ and $\sigma_j(\xi)$ by:

$$p_j(\xi) = p_0(2^{-j}\xi),$$

and

$$\sigma_j(\xi) = \sigma_0(2^{-j}\xi).$$

An immediate consequence of this construction is that

$$\sum_j p_j(\xi) = 1, \tag{2.1.1}$$

i.e. we have a partition of unity into functions $p_j(\xi)$.

We define Fourier multipliers P_j and S_j (on $L^2(\mathbb{R}^3)$) with their symbols $p_j(\xi)$ and $\sigma_j(\xi)$ respectively, i.e.

$$\hat{P_j f}(\xi) = p_j(\xi)\hat{f}(\xi),$$

$$\hat{S_j f}(\xi) = \sigma_j(\xi)\hat{f}(\xi).$$

The operators P_j and S_j are usually called Littlewood-Paley operators. We see that $P_j = S_j - S_{j-1}$ by this construction. In other words, the S_j's could be thought of as the partial sums of the P_j's.

As a consequence of (2.1.1) we have the Littlewood-Paley decomposition

$$f = \sum_j P_j f,$$

for all $f \in L^2(\mathbb{R}^3)$.

Intuitively speaking, we see that P_j is like a projection onto frequencies in the annulus $|\xi| \sim 2^j$, and S_j is like a projection onto frequencies in the ball $|\xi| \lesssim 2^j$. However we remark that P_j's and S_j's are not exactly projection operators. More precisely, it is not true that $(P_j)^2 = P_j$, nor it is true that $(S_j)^2 = S_j$. However we observe that

$$S_{j+2}P_j = P_j. \tag{2.1.2}$$

On the other hand, we have

$$S_{j-2}P_j = 0. \tag{2.1.3}$$

Thus (2.1.2) and (2.1.3) motivates the definition of operators $\tilde{P}_j = \sum_{k=-2}^{2} P_{j+k}$. We can analogously define the symbols $\tilde{p}_j = \sum_{k=-2}^{2} p_{j+k}$. We observe that

$$\tilde{P}_j P_j = P_j, \tag{2.1.4}$$

since $\tilde{P}_j$ is the sum of all Littlewood-Paley projections the support of whose symbols intersects the support of $p_j(\xi)$. We shall use (2.1.4) often to manipulate Littlewood-Paley operators. As an illustration of such a technique let us prove the Bernstein's inequality for Littlewood-Paley operators in $\mathbb{R}^3$:

Proposition 2.1.1

$$||P_j f||_{L^\infty} \lesssim 2^{\frac{3j}{2}} ||P_j f||_{L^2}.$$

Proof Let $\Phi_j = \check{(\tilde{p}_j)}$. Then (since we are working in $\mathbb{R}^3$) we have

$$\Phi_j(x) = 2^{3j}\Phi_0(2^j x).$$

Now $\tilde{P}_j f = \Phi_j * f$ so by Young's inequality, we have

$$||P_j f||_{L^\infty} = ||\tilde{P}_j P_j f||_{L^\infty} \leq ||\Phi_j||_{L^2} ||P_j f||_{L^2},$$

which is finite since Φ_j is a Schwartz function. But by the relation between Φ_j and Φ_0, we have

$$||\Phi_j||_{L^2} = 2^{\frac{3j}{2}} ||\Phi_0||_{L^2},$$

which proves the proposition. ■

Now we go to the "space-side" representation for the Littlewood-Paley operator P_j. Let $\phi_j = \check{(}p_j)$. Then $\phi_j(x) = 2^{3j}\phi_0(2^j x)$, and we have:

$$\begin{aligned} P_j f(x) &= (\phi_j * f)(x) \\ &= \int f(x-y)\phi_j(y)dy \\ &= \int f(x - 2^{-j}y)\phi_0(y)dy. \end{aligned} \tag{2.1.5}$$

We see that (2.1.5) and Minkowski's inequality imply:

$$\begin{aligned} ||P_j f||_{L^p} &\leq \int ||f(x-2^{-j}y)||_{L^p} |\phi_0(y)| dy \\ &\lesssim ||f||_{L^p}, \end{aligned}$$

for $1 \leq p \leq \infty$. This together with the triangle inequality implies

$$\sup_j ||P_j f||_{L^p} \lesssim ||f||_{L^p} \lesssim \sum_j ||P_j f||_{L^p}. \tag{2.1.6}$$

The inequality (2.1.6) is called a cheap Littlewood-Paley inequality. We will use it throughout the thesis. Material concerning the strong Littlewood-Paley inequality could be found in, for example, (10), (43), (45). We will not go in details of the strong Littlewood-Paley inequality, since we are not using it in this current work.

For writing the "space-side" representation of the Littlewood-Paley operator $\tilde{P}_j$ we use the notation introduced in the proof of Proposition 2.1.1. Then we can write $\tilde{P}_j$ as follows:

$$\begin{aligned} \tilde{P}_j f(x) &= (\Phi_j * f)(x) \\ &= \int f(x - 2^{-j}y)\Phi_0(y)dy. \end{aligned} \tag{2.1.7}$$

From (2.1.4) and (2.1.7) we obtain the following formula which relates the Littlewood-Paley piece $P_j f$ to itself:

$$P_j f(x) = \int P_j f(x - 2^{-j}y)\Phi_0(y)dy. \tag{2.1.8}$$

Now we are ready to see how differentiation acts on a Littlewood-Paley piece. Before formulating lemma precisely, we think a little bit what it means to take a derivative of a Littlewood-Paley piece. Informally speaking, if we look at the Fourier side we take derivative of a Littlewood-Paley piece $P_j f$

by multiplying its symbol by $2\pi i\xi$. However the symbol of the operator P_j, $p_j(\xi)$, is supported on $|\xi| \sim 2^j$. Therefore taking derivative of a Littlewood-Paley piece $P_j f$ should be related to multiplying $P_j f$ with 2^j. More precisely:

Lemma 2.1.2 *Let j be an integer. Then*

$$||\nabla P_j f||_{L^p} \sim 2^j ||P_j f||_{L^p},$$

for all $1 \leq p \leq \infty$.

Proof We present the proof following (45). By differentiating (2.1.8) we obtain:

$$\nabla P_j f = \int \nabla_x P_j f(x - 2^{-j} y)\Phi_0(y)dy,$$

which by integration by parts implies

$$\nabla P_j f = 2^j \int P_j f(x - 2^{-j} y)\nabla\Phi_0(y)dy. \tag{2.1.9}$$

Now we apply Minkowski's inequality on (2.1.9) and obtain

$$||\nabla P_j f||_{L^p} \leq 2^j ||P_j f||_{L^p},$$

since $\nabla\Phi_0$ is a Schwartz function.

Now we are left to prove

$$2^j||P_jf||_{L^p} \lesssim ||\nabla P_jf||_{L^p}. \tag{2.1.10}$$

In order to do that we recall the following property of the Fourier transform

$$\widehat{\nabla P_jf}(\xi) = 2\pi i\xi\widehat{P_jf}(\xi),$$

i.e.

$$\widehat{P_jf}(\xi) = \frac{1}{2\pi i\xi}\widehat{\nabla P_jf}(\xi),$$

and therefore

$$\widehat{P_jf}(\xi) = \frac{\tilde{p}_j(\xi)\ \xi\ \widehat{\nabla P_jf}(\xi)}{2\pi i|\xi|^2}. \tag{2.1.11}$$

Now by taking the inverse Fourier transform of (2.1.11) we have

$$\begin{aligned} P_jf &= \int \check{(}\frac{\tilde{p}_j(\xi)\ \xi}{2\pi i|\xi|^2})(y)\ \nabla P_jf(x-y)dy \\ &= \int \check{(}\frac{\tilde{p}_0(\frac{\xi}{2^j})\ \frac{\xi}{2^j}}{2\pi i|\frac{\xi}{2^j}|^2})(y)\ \nabla P_jf(x-y)dy. \end{aligned} \tag{2.1.12}$$

By changing a variable, (2.1.12) implies

$$2^jP_jf = \int \check{(}\frac{\tilde{p}_0(\xi)\ \xi}{2\pi i|\xi|^2})(y)\ \nabla P_jf(x-2^{-j}y)dy. \tag{2.1.13}$$

We notice that $\check{(\frac{\tilde{p}_0(\xi)\,\xi}{2\pi i|\xi|^2})}$ itself is a Schwartz function. Therefore we obtain (2.1.10) from applying Minkowski's inequality to (2.1.13). ■

2.2 Sobolev embedding theorem

Let $\mathbf{k} = (k_1, k_2, ..., k_n)$ be a multi-index, where k_i, $i = 1, ..., n$ are non-negative integers and $|\mathbf{k}| = k_1 + k_2 + ... + k_n$. Then for $f : \mathbb{R}^n \to \mathbb{C}$ we define

$$\nabla^{\mathbf{k}} := \frac{\partial^{|\mathbf{k}|} f}{\partial x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}}.$$

For $1 \leq p < \infty$ and $s \geq 0$ an integer, we define the Sobolev space $W^{s,p}(\mathbb{R}^n)$ as a space of all functions f such that f and its derivatives up to order s are in $L^p(\mathbb{R}^n)$. The norm in this space is introduced by

$$||f||_{W^{s,p}(\mathbb{R}^n)} := \sum_{k=1}^{s} ||\nabla^k f||_{L^p(\mathbb{R}^n)}. \tag{2.2.1}$$

We see that p stands for integrability, while s stands for differentiability. In particular when $p = 2$ the space $W^{s,2}$ is denoted by H^s, and it is a Hilbert space.

The above definition of the Sobolev space $W^{s,p}$ can be extended so that it is valid for $s \in \mathbb{R}$. Before we do that, let us recall the notion of a fractional derivative. A property of the Fourier transform gives

$$\widehat{(-\Delta)f}(\xi) = 4\pi^2|\xi|^2\hat{f}(\xi).$$

This gives motivation to define the operator $|\nabla|$ as the square root of the operator $-\Delta$ by

$$\widehat{|\nabla| f}(\xi) := 2\pi|\xi|\hat{f}(\xi).$$

Also for any real s, we define the fractional derivative operator $|\nabla|^s$ by

$$\widehat{|\nabla|^s f}(\xi) := (2\pi|\xi|)^s \hat{f}(\xi).$$

In a similar way we define the modified fractional derivative operator $\langle\nabla\rangle^s$ by

$$\widehat{\langle\nabla\rangle^s f}(\xi) := \langle 2\pi\xi\rangle^s \hat{f}(\xi),$$

where the symbol $\langle\cdot\rangle$ is known as the Japanese bracket and is defined by $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$. Notice that for higher frequencies the operator $\langle\nabla\rangle^s$ behaves the same as the operator $|\nabla|^s$, while for the frequencies $|\xi| \lesssim 1$ the operator $\langle\nabla\rangle^s$ acts as an identity.

Now we are ready to introduce the Sobolev space $W^{s,p}$ for $s \in \mathbb{R}$ and $1 \leq p < \infty$ as a space of functions f such that $(1+|\xi|)^{s/2}\hat{f}(\xi) \in L^p$. In particular, when $p = 2$ the norm in the space H^s is introduced by

$$||f||_{H^s(\mathbb{R}^n)} := (\int_{\mathbb{R}^n} |\langle\xi\rangle^s \hat{f}(\xi)|^2 d\xi)^{\frac{1}{2}}.$$

Here we shall state and prove a version of Sobolev embedding theorem in $\mathbb{R}^3$.

Theorem 2.2.1 *Let $p > 2$. There exist an $\epsilon > 0$ and a constant $C(p,\epsilon)$ such that:*

$$||f||_{L^p} \leq C(p,\epsilon)||f||_{H^{3(\frac{1}{2}-\frac{1}{p})+\epsilon}}.$$

Proof We decompose f using Littlewood-Paley operators as $f = \sum_{j=-\infty}^{\infty} P_j f$.

From the definition of H^α-norm we see that

$$||P_j f||_{L^2} \sim 2^{-\alpha j}||P_j f|_{H^\alpha},$$

and therefore

$$||P_j f||_{L^2} \lesssim 2^{-\alpha j}||f||_{H^\alpha}. \tag{2.2.2}$$

On the other hand, Bernstein's inequality guarantees

$$||P_j f||_{L^\infty} \lesssim 2^{\frac{3j}{2}}||P_j f||_{L^2},$$

which by (2.2.2) implies

$$||P_j f||_{L^\infty} \lesssim 2^{\frac{3j}{2}}2^{-\alpha j}||f||_{H^\alpha}. \tag{2.2.3}$$

However by Hölder's inequality we have

$$||f||_{L^p} \leq ||f||_{L^\infty}^{\frac{p-2}{p}} ||f||_{L^2}^{\frac{2}{p}},$$

and in particular

$$||P_j f||_{L^p} \leq ||P_j f||_{L^\infty}^{\frac{p-2}{p}} ||P_j f||_{L^2}^{\frac{2}{p}}. \tag{2.2.4}$$

Notice that (2.2.4) together with bounds obtained in (2.2.2) and (2.2.3) implies

$$\begin{aligned} ||P_j f||_{L^p} &\lesssim (2^{\frac{3j}{2}} 2^{-\alpha j} ||f||_{H^\alpha})^{\frac{p-2}{p}} (2^{-\alpha j} ||f||_{H^\alpha})^{\frac{2}{p}} \\ &= 2^{j[(\frac{3}{2}-\alpha)\frac{p-2}{p} - \alpha\frac{2}{p}]} ||f||_{H^\alpha}. \end{aligned} \tag{2.2.5}$$

Now for $\alpha > 3(\frac{1}{2} - \frac{1}{p})$ we can sum (2.2.5) over j, and the theorem is proved. ■

2.3 Frequency trichotomy

Here we shall use the Littlewood-Paley operators to analyze the pointwise product of two functions f and g. Let us see what we can say about $P_j(fg)$.

First we shall represent f and g using Littlewood-Paley operators as follows

$$f = \sum_{k_1} P_{k_1} f,$$

$$g = \sum_{k_2} P_{k_2} g,$$

and therefore

$$P_j(fg) = \sum_{k_1,k_2} P_j(P_{k_1} f P_{k_2} g). \tag{2.3.1}$$

Since $\widehat{P_{k_i} f}(\xi)$ is supported in the annulus $A_{k_i} := \{\xi : 2^{k_i-1} \le |\xi| \le 2^{k_i+1}\}$, for $i = 1, 2$ we know that the product $P_{k_1} f \ P_{k_2} g$ has Fourier support in the sum of the annuli A_{k_1} and A_{k_2}. On the other hand, in order that the double sum in (2.3.1) is nonzero the sum $A_{k_1} + A_{k_2}$ should intersect the support of $\widehat{P_j f}(\xi)$, which is the annulus $\{\xi : 2^{j-1} \le |\xi| \le 2^{j+1}\}$.

Thus (2.3.1) can be rewritten as follows:

$$P_j(fg) = H_{j,lh} + H_{j,hl} + H_{j,ll} + H_{j,hh},$$

where the low-high part is given by

$$H_{j,lh} = P_j\Big(\sum_{k<j-4} (P_k f) \cdot \tilde{P}_j g\Big),$$

the high-low part is given by

$$H_{j,hl} = P_j\Big(\sum_{k<j-4} (\tilde{P}_j f) \cdot P_k g\Big),$$

the low-low part is given by

$$H_{j,ll} = P_j(\sum_{j-4\leq k\leq j+4} (P_k f) \cdot \tilde{P}_k g) + P_j(\sum_{j-4\leq k\leq j+4} (\tilde{P}_k f) \cdot P_k g),$$

and the high-high part is given

$$H_{j,hh} = P_j(\sum_{k>j+4} (P_k f) \cdot \tilde{P}_k g) + P_j(\sum_{k>j+4} (\tilde{P}_k f) \cdot P_k g).$$

Therefore there are four interactions which are nonzero. However the low-low part is usually included either in the high-low part, or in the low-high part, and such a decomposition of $P_j(fg)$ is known as the Littlewood-Paley trichotomy. The four sums $H_{j,lh}$, $H_{j,hl}$, $H_{j,ll}$, $H_{j,hh}$ are called paraproducts too. For more details on paraproducts see, for example, (47).

CHAPTER 3

CLASSICAL EXAMPLES

3.1 Introduction

In this chapter we present some classical examples of fluid dynamics by discussing how they could be seen from the perspective of Littlewood-Paley operators. First we give a summary of the famous paper of Beale-Kato-Majda (1) which gives a criterion for loss of regularity for solutions of the Euler equations. After recalling main ideas of (1), we use Littlewood-Paley operators to prove a logarithmic inequality playing an important role in (1).

Then we revisit (1) again. We employ Littlewood-Paley operators to prove a theorem similar to one of Beale-Kato-Majda. The proof of such a theorem illustrates specific roles of high and low frequencies.

We also show how Sobolev embedding theorem (which was proved in chapter 1 by using Littlewood-Paley operators too) could be applied to prove global strong solvability for the Navier-Stokes equations with enough dissipation.

The ideas in the use of Littlewood-Paley operators combined with pseudodifferential operators were developed by Bony (3) and Meyer (33). The method of (3) found various applications in styding nonlinear PDEs, since according to paraproduct representation, the method is sensitive to describing behavior of the nonlinear term. In the context of applications of Littlewood-Paley operators to studying equations of fluid motion, see for example, (8), (7), (6), (49), (50).

In this chapter we illustrate some aspects of using Littlewood-Paley operators in studying the Euler and the Navier-Stokes equations. In particular, by applying Littlewood-Paley operators P_j onto the velocity field $u(x,t)$, we localize the velocity field to range of frequencies around 2^j. The question which arises is why it is enough to localize to frequency ranges. In the present work we estimate Lebesgue and Sobolev norms of some fluid quantities. The cheap Littlewood-Paley inequality, which was discussed in the previous chapter, tells us that we can reveal information about certain Lebesgue or Sobolev norms of a function just by knowing the norms of Littlewood-Paley pieces, rather then calculating Fourier coefficients for each different frequency. This is only one advantage of Littlewood-Paley operators. However more important reason for their succesfull applications for studying the Euler and the Navier-Stokes equations is that the method is well suited for analysing the nonlinear term. The worst term in both the Euler and the Navier-Stokes equations is the nonlinear term $u \cdot \nabla u$, which by using Littlewood-Paley operators can be represented using paraproducts. In such a way one isolates certain frequency ranges and their interactions which are responsible for possible growth of solutions to the equations of fluid motion.

The chapter is organized as follows. In section 3.2 we recall the work of Beale-Kato-Majda (1). In section 3.3 we use Littlewood-Paley operators to prove a theorem similar to one in (1). In section 3.4 we prove a global strong solvability for the Navier-Stokes equations with enough dissipation.

3.2 Theorem of Beale-Kato-Majda

Let $u = u(x,t)$ be the velocity field, and $p = p(x,t)$ the pressure. Then the Euler equations are:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0, \tag{3.2.1}$$

$$\nabla \cdot u = 0. \tag{3.2.2}$$

It is an outstanding problem of fluid dynamics to find out if solutions of the Euler equations form singularities in finite time. The answer is "no" in the case of Euler equations in dimension two. But in three-dimensional space the question is still open.

However the local existence theorem for Euler's equations is known, and we will state it here according to the statement given in (1):

Theorem 3.2.1 *Suppose an initial velocity field u_0 is specified in $H^s(\mathbb{R}^3)$, $s \geq 3$, with $||u_0||_{H^3(\mathbb{R}^3)} \leq N_0$, for some $N_0 > 0$. Then there exists $T_0 > 0$, depending only on N_0, so that* (3.2.1), (3.2.2) *have a solution in the class*

$$u \in C([0,T]; H^s(\mathbb{R}^3)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^3)) \tag{3.2.3}$$

at least for $T = T_0(N_0)$.

This theorem does not say if solutions actually lose their regularity.

Analytical and numerical results suggest the connection between the accumulation of vorticity $\omega = \nabla \times u$ and development of finite time singularities for the three-dimensional Euler equations. Beale, Kato and Majda in (1) made this connection mathematically rigorous by proving the following theorem:

Theorem 3.2.2 *Let u be a solution of Euler's equations* (3.2.1), (3.2.2), *and suppose there is a time T_* such that the solution cannot be continued in the class (3.2.3) to $T = T_*$. Assume that T_* is the first such time. Then*

$$\int_0^{T_*} ||\omega(t)||_{L^\infty} dt = \infty,$$

and in particular

$$\limsup_{t \longrightarrow T_*} ||\omega(t)||_{L^\infty} = \infty.$$

Thus Theorem 3.2.2 gives a criterion for loss of regularity in the sense that if the solution fails to be regular past a certain time, then the vorticity must be unbounded.

Here we will present main ideas of the proof of Theorem 3.2.2 following (1). The proof is by contradiction. Restricted to this chapter $H^s(\mathbb{R}^3)$ is denoted by H^s, and $L^p(\mathbb{R}^3)$, with $1 \leq p \leq \infty$ is denoted by L^p.

Let us assume

$$\int_0^{T_*} ||\omega(t)||_{L^\infty} dt < \infty. \tag{3.2.4}$$

Then we will show that

$$||u(t)||_{H^s} \leq C_0, \text{ for all } t < T_*. \tag{3.2.5}$$

However (3.2.5) leads to contradiction, because if (3.2.5) could be true then by the local existence theorem, Theorem 3.2.1, we would be able to extend the original solution past time T_* contrary to the choice of T_*.

The main ingredients of the proof are the following three steps:

Step 1 We bound $||\omega(t)||_{L^2}$ by $||\omega(t)||_{L^\infty}$. In order to to that we start from the vorticity equation:

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \omega \cdot \nabla u. \tag{3.2.6}$$

Then we pair in $L^2(\mathbb{R}^3)$ the equation (3.2.6) with ω and obtain:

$$\frac{1}{2}\frac{d}{dt}||\omega||_{L^2}^2 = \langle \omega \cdot \nabla u, \omega \rangle. \tag{3.2.7}$$

To simplify notation we define $m(t)$ by $m(t) = ||\omega(t)||_{L^\infty}$. Now from (3.2.7) by using Cauchy-Schwartz inequality, Hölder inequality and a relation between u and ω we obtain the following estimate:

$$\frac{1}{2}\frac{d}{dt}||\omega||_{L^2}^2 \le Cm(t)||\omega||_{L^2}^2,$$

which by Gronwall's inequality implies

$$||\omega(T)||_{L^2} \le ||\omega(0)||_{L^2} e^{C\int_0^T m(t)dt}. \tag{3.2.8}$$

Step 2 Now we bound $||u||_{H^s}$ by $||\nabla u||_{L^\infty}$. In particular, we apply the operator D^α to equations (3.2.1), (3.2.2), where α is a multi-index with $|\alpha| \le s$.

Having introduced

$$v = D^{\alpha} u,$$

$$q = D^{\alpha} p,$$

and

$$F = D^{\alpha}(u \cdot \nabla u) - u \cdot D^{\alpha}(\nabla u),$$

we obtain:

$$\frac{\partial v}{\partial t} + u \cdot \nabla v + \nabla q = -F. \tag{3.2.9}$$

Then we pair the equation (3.2.9) with v and obtain:

$$\frac{1}{2}\frac{d}{dt}||v||_{L^2}^2 = -\langle F, v\rangle. \tag{3.2.10}$$

By using Gagliardo-Nirenberg inequality we obtain a bound on $||F||_{L^2}$. Then after summing over α in (3.2.10) and applying Gronwall's inequality we have:

$$||u||_{H^s}^2 \leq ||u(0)||_{H^s}^2 e^{C\int_0^T ||\nabla u||_{L^\infty} dt}. \tag{3.2.11}$$

Step 3 We state the inequality:

$$||\nabla u||_{L^\infty} \leq C\{1 + (1 + \log ||u||_{H^3})||\omega||_{L^\infty} + ||\omega||_{L^2}\}, \tag{3.2.12}$$

which gives bound on $||\nabla u||_{L^\infty}$ in terms of $||\omega||_{L^\infty}$ and $||\omega||_{L^2}$. The proof of (3.2.12) presented in (1) uses the Biot-Savart Law and singular integrals approach.

Now we combine (3.2.8), (3.2.11), and (3.2.12) and in such a way we prove (3.2.5).

Here we point out the the inequality (3.2.12) could be proved using Littlewood-Paley operators too. We present that proof in the following lemma:

Lemma 3.2.3 *Under above conditions we have:*

$$||\nabla u||_{L^\infty} \leq C\{1 + (1 + \log ||u||_{H^3})||\omega||_{L^\infty} + ||\omega||_{L^2}\}.$$

Proof From the Biot-Savart Law

$$u(x) = -\frac{1}{4\pi} \int \frac{x-y}{|x-y|^3} \times w(y)dy,$$

we conclude that ∇u is a singular integral operator of ω, and we will denote that operator by $T\omega$.

Now we shall use Littlewood-Paley operators to decompose $T\omega$ as

$$T\omega = \sum_{k=-\infty}^{\infty} P_k T\omega.$$

However notice that $P_k T\omega$ is no more singular and we have:

$$||P_k T\omega||_{L^\infty} \lesssim ||P_k \omega||_{L^\infty}. \tag{3.2.13}$$

Now we develop three different upper bounds on $||P_k\omega||_{L^\infty}$. At the end we will compare them and use each when it is most efficient.

First let us notice that by using (3.2.13), Bernstein's inequality and definition of H^2 norm we have:

$$\begin{aligned}
||P_k T\omega||_{L^\infty} &\lesssim ||P_k\omega||_{L^\infty} \\
&\leq 2^{\frac{3k}{2}}||P_k\omega||_{L^2} \qquad (3.2.14)\\
&\leq 2^{-\frac{k}{2}}||P_k\omega||_{H^2} \\
&\leq 2^{-\frac{k}{2}}||\omega||_{H^2}, \qquad (3.2.15)
\end{aligned}$$

where the last inequality comes from the cheap Littlewood-Paley inequality.

However for k's such that

$$2^{-\frac{k}{3}}||\omega||_{H^2} < ||\omega||_{L^\infty}, \tag{3.2.16}$$

(3.2.15) implies:

$$||P_k T\omega||_{L^\infty} \lesssim 2^{-\frac{k}{6}}||\omega||_{L^\infty}. \tag{3.2.17}$$

On the other hand for k's such that

$$2^{-\frac{k}{3}}||\omega||_{H^2} \geq ||\omega||_{L^\infty},$$

we simple notice that (3.2.13) together with the cheap Littlewood-Paley inequality implies the bound:

$$||P_k T\omega||_{L^\infty} \lesssim ||\omega||_{L^\infty}. \tag{3.2.18}$$

For negative $k's$ we will use the following consequence of (3.2.14) and the cheap Littlewood-Paley inequality:

$$||P_k T\omega||_{L^\infty} \lesssim 2^{\frac{3k}{2}}||\omega||_{L^2}. \tag{3.2.19}$$

Notice that (3.2.16) means $k \gtrsim \log \frac{||\omega||_{H^2}}{||\omega||_{L^\infty}}$. Now we combine bounds (3.2.17), (3.2.18) and (3.2.19). More precisely we sum over k and use:

- (3.2.17) for $k \geq \log \frac{||\omega||_{H^2}}{||\omega||_{L^\infty}}$,
- (3.2.18) for $k < \log \frac{||\omega||_{H^2}}{||\omega||_{L^\infty}}$ and
- (3.2.19) for negative k's. Thus we obtain:

$$||T\omega||_{L^\infty} \leq (1 + \log \frac{||\omega||_{H^2}}{||\omega||_{L^\infty}})||\omega||_{L^\infty} + ||\omega||_{L^2}. \tag{3.2.20}$$

If $||\omega||_{L^\infty} \geq 1$ then (3.2.20) implies

$$||T\omega||_{L^\infty} \leq (1 + \log ||\omega||_{H^2})||\omega||_{L^\infty} + ||\omega||_{L^2}, \tag{3.2.21}$$

while if $||\omega||_{L^\infty} < 1$ then

$$(\log \frac{||\omega||_{H^2}}{||\omega||_{L^\infty}}) \cdot ||\omega||_{L^\infty} \leq 1 + (\log ||\omega||_{H^2})||\omega||_{L^\infty},$$

which together with (3.2.20) implies the claim of the lemma. ■

3.3 Theorem of Beale-Kato-Majda from Littlewood-Paley perspective

Here we state a theorem which gives a criterion for loss of regularity for the solutions of the Euler equations. We prove the theorem using Littlewood-Paley operators. More specifically we prove:

Theorem 3.3.1 *Let u be a solution of Euler's equations (3.2.1), (3.2.2), and suppose there is a time T_* such that the solution cannot be continued in the class (3.2.3) to $T = T_*$. Assume that T_* is the first such time. Then*

$$||\omega(T_*)||_{L^\infty} = \infty.$$

Proof Let us assume

$$||\omega(t)||_{L^\infty} dt < \infty, \text{ for all } t \in [0, T_*]. \tag{3.3.1}$$

Then we will show that

$$||u(t)||_{H^s} \leq C_0, \text{ for all } t < T_*. \tag{3.3.2}$$

However (3.3.2) leads to contradiction, because if (3.3.2) could be true then by the local existence theorem, Theorem 3.2.1, we would be able to extend the original solution past time T_* contrary to the choice of T_*.

Let us consider the vorticity equation.

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \omega \cdot \nabla u. \tag{3.3.3}$$

Now we compute the $L^2(\mathbb{R}^3)$ pairing of the equation (3.3.3) with $P_j^2 \omega$. We obtain:

$$\frac{1}{2}\frac{d}{dt}||P_j\omega||_{L^2}^2 = -\langle P_j(u \cdot \nabla\omega), P_j\omega\rangle + \langle P_j(\omega \cdot \nabla u), P_j\omega\rangle \tag{3.3.4}$$

We shall estimate the terms on the right hand side of (3.3.4).

Lemma 3.3.2 *Under above conditions we have:*

$$||P_j(\omega \cdot \nabla u)||_{L^2} \lesssim ||P_j\omega||_{L^2}(\sum_{k\leq j+4} ||P_k\omega||_{L^\infty}) + \sum_{k>j+4} 2^{\frac{3j}{2}}||P_k\omega||_{L^2}^2.$$

Proof We rewrite $P_j(\omega \cdot \nabla u)$ using “trichotomy”:

$$P_j(\omega \cdot \nabla u) = H_{j,lh} + H_{j,hl} + H_{j,ll} + H_{j,hh},$$

where the low-high part is given by

$$H_{j,lh} = P_j(\sum_{k<j-4} (P_k\omega) \cdot \tilde{P}_j \nabla u),$$

the high-low part is given by

$$H_{j,hl} = P_j(\sum_{k<j-4} (\tilde{P}_j\omega) \cdot P_k \nabla u),$$

the low-low part is given by

$$H_{j,ll} = P_j(\sum_{j-4\leq k\leq j+4} (P_k\omega) \cdot \tilde{P}_k \nabla u) + P_j(\sum_{j-4\leq k\leq j+4} (\tilde{P}_k\omega) \cdot P_k \nabla u),$$

and the high-high part is given

$$H_{j,hh} = P_j(\sum_{k>j+4} (P_k\omega) \cdot \tilde{P}_k \nabla u) + P_j(\sum_{k>j+4} (\tilde{P}_k\omega) \cdot P_k \nabla u).$$

We bound the low-high part by using a cheap Littlewood-Paley inequality, triangle inequality and Hölder inequality:

$$\begin{aligned}
||H_{j,lh}||_{L^2} &\leq ||\sum_{k<j-4} (P_k\omega) \cdot \tilde{P}_j \nabla u||_{L^2} \\
&\leq \sum_{k<j-4} ||(P_k\omega) \cdot \tilde{P}_j \nabla u||_{L^2} \\
&\leq \sum_{k<j-4} ||(P_k\omega)||_{L^\infty} \cdot ||\tilde{P}_j \nabla u||_{L^2} \\
&\lesssim ||P_j\omega||_{L^2}(\sum_{k<j-4} ||P_k\omega||_{L^\infty}).
\end{aligned}$$

In a similar spirit, by using a cheap Littlewood-Paley inequality, triangle inequality and Hölder inequality, we bound the high-low term as:

$$\begin{aligned}
||H_{j,hl}||_{L^2} &\leq ||\sum_{k<j-4} (\tilde{P}_j\omega)\cdot P_k\nabla u||_{L^2} \\
&\leq \sum_{k<j-4} ||(\tilde{P}_j\omega)\cdot P_k\nabla u||_{L^2} \\
&\leq \sum_{k<j-4} ||(\tilde{P}_j\omega)||_{L^2}\cdot ||P_k\nabla u||_{L^\infty} \\
&\lesssim ||P_j\omega||_{L^2}(\sum_{k<j-4} ||P_k\omega||_{L^\infty}).
\end{aligned}$$

Also by using a cheap Littlewood-Paley inequality, triangle inequality and Hölder inequality, we obtain the following bound for the low-low part:

$$||H_{j,ll}||_{L^2} \lesssim ||P_j\omega||_{L^2}(\sum_{j-4\leq k\leq j+4} ||P_k\omega||_{L^\infty}).$$

In order to bound the high-high part we will first recall that for any functions f, g: $||P_j(P_kf\cdot P_kg)||_{L^2} = |\langle P_j(P_kf\cdot P_kg), h\rangle|$, where $||h||_{L^2}=1$. Thus by using Cauchy-Schwartz inequality and Proposition 2.1.1 we obtain:

$$\begin{aligned}
||P_j(P_kf\cdot P_kg)||_{L^2} &\leq |\langle P_kf\cdot P_kg, P_jh\rangle| \\
&\leq ||P_jh||_{L^\infty}||P_kf||_{L^2}||P_kg||_{L^2} \\
&\leq 2^{\frac{3j}{2}}||h||_{L^2}||P_kf||_{L^2}||P_kg||_{L^2} \\
&= 2^{\frac{3j}{2}}||P_kf||_{L^2}||P_kg||_{L^2}. \qquad (3.3.5)
\end{aligned}$$

Now we apply (3.3.5) with f and g equal to ω and ∇u respectively. In such a way we obtain a bound for the high-high part as:

$$||H_{j,hh}||_{L^2} \lesssim \sum_{k>j+4} 2^{\frac{3j}{2}} ||P_k \omega||_{L^2}^2.$$

Combining bounds for $H_{j,lh}$, $H_{j,hl}$ and $H_{j,hh}$ the claim is proved. ∎

In a similar way we obtain the following type of bound for the first term of the right hand side of (3.3.4):

Lemma 3.3.3 *Under above conditions we have:*

$$||\langle P_j(u \cdot \nabla \omega), P_j \omega \rangle||_{L^2} \lesssim ||P_j \omega||_{L^2} (\sum_{k \leq j+4} ||P_k \omega||_{L^\infty}) + \sum_{k>j+4} 2^{\frac{3j}{2}} ||P_k \omega||_{L^2}^2.$$

Thus by Lemma 3.3.2 and Lemma 3.3.3, the expression (3.3.4) implies:

$$\frac{d}{dt} ||P_j \omega||_{L^2} \lesssim ||P_j \omega||_{L^2} (\sum_{k \leq j+4} ||P_k \omega||_{L^\infty}) + \sum_{k>j+4} 2^{\frac{3j}{2}} ||P_k \omega||_{L^2}^2. \tag{3.3.6}$$

By using Proposition 2.1.1 (Bernstein's inequality), we see that (3.3.6) implies:

$$\frac{d}{dt} ||P_j \omega||_{L^2} \lesssim ||P_j \omega||_{L^2} (\sum_{k \leq j+4} 2^{\frac{3k}{2}} ||P_k \omega||_{L^2}) + \sum_{k>j+4} 2^{\frac{3j}{2}} ||P_k \omega||_{L^2}^2. \tag{3.3.7}$$

Now we multiply the inequality (3.3.7) by $2^{\frac{3j}{2}}$ and we choose a sequence $\{\omega_k\}$ in such a way that

$$2^{\frac{3k}{2}} ||P_k \omega||_{L^2} < \omega_k, \text{ as long as } \omega_k < 1.$$

Then our system (3.3.7) is majorized by the system:

$$\frac{d\omega_j}{dt} = \omega_j \sum_{k\leq j} \omega_k + \sum_{k>j} \omega_k^2, \text{ when } \omega_j < 1, \tag{3.3.8}$$

and

$$\omega_j = 1 \text{ once } \omega_j \text{ reaches } 1, \tag{3.3.9}$$

with the initial condition

$$\omega_j(0) = 2^{-\alpha j}. \tag{3.3.10}$$

Now we will prove a simple lemma which will enable us to majorize the system (3.3.8)-(3.3.10).

Lemma 3.3.4 *If $\{\omega_l\}$ is a sequence of solutions to the system* (3.3.8)-(3.3.10) *then*

$$\omega_k < \omega_j,$$

when $k > j$, i.e. the sequence $\{\omega_l\}$ is a decreasing sequence.

Proof It suffices to prove $\omega_{j+1} < \omega_j$.

Note that initially we have:

$$\omega_{j+1}(0) = 2^{-\alpha(j+1)} < 2^{-\alpha j} = \omega_j(0).$$

Thus it is enough to prove:

$$\frac{d\omega_{j+1}}{dt} < \frac{d\omega_j}{dt}.$$

Since ω_j and ω_{j+1} are solutions to the system (3.3.8)-(3.3.10) we can calculate $\frac{d\omega_{j+1}}{dt} - \frac{d\omega_j}{dt}$ and we obtain:

$$\frac{d(\omega_j - \omega_{j+1})}{dt} = (\sum_{k\leq j} \omega_k)(\omega_j - \omega_{j+1}),$$

and therefore

$$(\omega_j - \omega_{j+1})(t) = (\omega_j - \omega_{j+1})(0)\ e^{\int_0^t \sum_{k\leq j} \omega_k(\tau)d\tau},$$

which implies

$$(\omega_j - \omega_{j+1})(t) > 0,$$

and the lemma is proved. ■

Now from Lemma 3.3.4 we see that the system (3.3.8)-(3.3.10) can be majorized by the system:

$$\frac{d\omega_j}{dt} = (\sum_{k=1}^{\infty} \omega_k)\omega_j \tag{3.3.11}$$

and

$$\omega_j = 1 \text{ once } \omega_j \text{ reaches } 1, \tag{3.3.12}$$

with the initial condition

$$\omega_j(0) = 2^{-\alpha j}. \tag{3.3.13}$$

Notice that for the system (3.3.11) - (3.3.13) we always have

$$\omega_{j+1} = 2^{-\alpha}\omega_j \text{ as long as } \omega_j \text{ is not yet 1.} \tag{3.3.14}$$

Thus at the time t_1 when ω_1 becomes 1, by shifting the index we are almost in the situation we had at $t = 0$ because $\omega_2(t_1) = 2^{-\alpha}$. From this point on ω_1 is fixed at 1, so

$$\begin{aligned} \omega_1 &= 1. \\ \frac{d\omega_j}{dt} &= (1 + \sum_{k=2}^{\infty} \omega_k)\omega_j. \end{aligned}$$

At the m^{th} step we have:

$$\frac{d\omega_j}{dt} = (m + \sum_{k=m+1}^{\infty} \omega_k)\omega_j. \tag{3.3.15}$$

Now we prove the following statement:

Lemma 3.3.5 *Let* $\{\omega_l\}$ *be a sequence of solution to the system* (3.3.11) - (3.3.13)*. If* ω_1, ω_2, ..., ω_m *have already become equal to 1, then* ω_{m+1} *will go from* $2^{-\alpha}$ *to* 1 *in time like* $\frac{1}{m}$.

Proof Since

$$
\begin{aligned}
\omega_1 &= 1, \\
&\ldots \\
\omega_m &= 1,
\end{aligned}
$$

by recalling (3.3.14) we have

$$
\begin{aligned}
\sum_{k=m+1}^{\infty} \omega_k &= \omega_{m+1} + 2^{-\alpha}\omega_{m+1} + 2^{-2\alpha}\omega_{m+1} + \ldots \\
&\leq 1 + 2^{-\alpha} + 2^{-2\alpha} + \ldots \\
&= C.
\end{aligned}
\tag{3.3.16}
$$

Since ω_{m+1} satisfies (3.3.15) we have

$$
\frac{d\omega_{m+1}}{dt} = (m + \sum_{k=m+1}^{\infty} \omega_k)\omega_{m+1},
$$

which together with (3.3.16) implies

$$
\frac{d\omega_{m+1}}{dt} \leq (m + C)\omega_{m+1}. \tag{3.3.17}
$$

We denote the time when ω_m becomes equal to 1 by t_m, and similarly we denote by t_{m+1} the time when ω_{m+1} becomes equal to 1. Now we integrate (3.3.17) on the time interval $[t_m, t_{m+1}]$ and conclude that $t_{m+1} - t_m$ is of order $\frac{1}{m}$. ∎

Now from Lemma 3.3.5 we conclude that w_{m+1} will have gone from $2^{-\alpha(m+1)}$ to 1 in time like

$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{m} = \log\ m,$$

and therefore for all $t < \log\ m$ we have

$$\omega_{m+1} \leq 2^{2^t} 2^{-\alpha(m+1)}. \tag{3.3.18}$$

In the other words for all $k > 2^t$ we have

$$\omega_k \leq 2^{2^t} 2^{-\alpha k}. \tag{3.3.19}$$

We are almost ready to conclude the proof of Theorem 3.3.1. In order to do that we will use the following two inequalities. The first inequality comes from the conservation of energy for the Euler equation and can be formulated as:

$$||P_k \omega||_{L^2} \leq 2^k, \tag{3.3.20}$$

and we will use it for $k \leq 2^t$. However for $k > 2^t$ we will use the inequality

$$2^{\frac{3k}{2}}||P_k\omega||_{L^2} \leq 2^{2^t}2^{-\alpha k}, \tag{3.3.21}$$

which follows from (3.3.19) by recalling the definition of ω_k.

Now we notice that (3.3.20) in terms of H^β-norm implies

$$||P_k\omega||_{H^\beta} \leq 2^{\beta k}2^k, \text{ for } k \leq 2^t, \tag{3.3.22}$$

while (3.3.21) implies

$$||P_k\omega||_{H^\beta} \leq 2^{(\beta-\alpha)k}2^{2^t}2^{-\frac{3k}{2}}, \text{ for } k > 2^t. \tag{3.3.23}$$

We sum $||P_k\omega||_{H^\beta}$ over k and by using (3.3.22), (3.3.23) and a cheap Littlewood-Paley inequality we obtain:

$$||\omega||_{H^\beta} \leq 2^t 2^{(\beta+1)2^t} + \sum_{k>2^t} 2^{(\beta-\alpha)k}2^{2^t}2^{-\frac{3k}{2}},$$

and therefore for $\beta < \alpha + \frac{3}{2}$

$$||u||_{H^{\beta+1}} \leq constant,$$

and the theorem is proved. ■

3.4 Global strong solvability for the Navier-Stokes equations with enough dissipation

In this section we present a proof of global strong solvability for the Navier-Stokes equations with enough dissipation. We could think about this proof as about an application of the Sobolev embedding theorem.

The Navier-Stokes equation with dissipation $(-\Delta)^{\alpha}$ is given by

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = -(-\Delta)^{\alpha} u, \tag{3.4.1}$$

where u is a time-dependent divergence free vector field in $\mathbb{R}^3$. One sets the initial condition

$$u(x, 0) = u_0(x) \tag{3.4.2}$$

where $u_0(x) \in C_c^{\infty}(\mathbb{R}^3)$.

The well known fact about this is expressed in Proposition 3.4.1. We learned the proof which we present in this section from Diego Cordoba.

Proposition 3.4.1 *If* $\alpha \geq \frac{5}{4}$, *one has global strong solvability for the system* (3.4.1), (3.4.2).

Proof We will use the standard notation that H^{β} denotes the L^2 Sobolev space over $\mathbb{R}^3$ with β derivatives.

We first recall the energy inequality obtained by pairing the equation (3.4.1) with u

$$\frac{1}{2}\frac{\partial(||u||_{L^2}^2)}{\partial t} = -||(-\Delta)^{\frac{\alpha}{2}} u||_{L^2}^2 \leq -||u||_{H^{\alpha}}^2 + ||u||_{L^2}^2. \tag{3.4.3}$$

From this, we obtain by integrating over time (and observing that the L^2 norm is always positive), that if the solution u remains smooth up to time T, we have the estimate

$$\int_0^T ||u||_{H^\alpha}^2 \, dt \lesssim (1+T). \tag{3.4.4}$$

We now pair (3.4.1) with $(-\Delta)u$ in order to estimate $\frac{\partial(||u||_{H^1}^2)}{\partial t}$. We obtain

$$\frac{1}{2}\frac{\partial(||(-\Delta)^{\frac{1}{2}}u||_{L^2}^2)}{\partial t} + \langle u \cdot \nabla u, (-\Delta)u\rangle = -||(-\Delta)^{\frac{\alpha+1}{2}}u||_{L^2}^2. \tag{3.4.5}$$

Clearly, we must estimate the nonlinear problem term

$$\langle u \cdot \nabla u, (-\Delta)u\rangle.$$

Using Hölder's inequality we have the following estimate

$$|\langle u \cdot \nabla u, (-\Delta)u\rangle| \leq ||u||_{L^p}||\nabla u||_{L^2}||(-\Delta)u||_{L^q}, \tag{3.4.6}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.

Now we apply Sobolev embedding theorem to the right-hand side of (3.4.6) and obtain

$$|\langle u \cdot \nabla u, (-\Delta)u\rangle| \leq ||u||_{H^{3(\frac{1}{2}-\frac{1}{p})}}||u||_{H^1}||u||_{H^{3(\frac{1}{2}-\frac{1}{q})+2}}. \tag{3.4.7}$$

Since we are assuming $\alpha \geq \frac{5}{4}$, we obtain from (3.4.7)

$$|\langle u \cdot \nabla u, (-\Delta) u \rangle| \leq ||u||_{H^\alpha} ||u||_{H^1} ||u||_{H^{\alpha+1}}. \tag{3.4.8}$$

Applying Cauchy-Schwartz, we get immediately

$$||u||_{H^\alpha} ||u||_{H^1} ||u||_{H^{\alpha+1}} \leq \delta ||u||^2_{H^{\alpha+1}} + \frac{1}{\delta} ||u||^2_{H^\alpha} ||u||^2_{H^1}.$$

Combining this with (3.4.5), we get

$$\frac{\partial ||u||^2_{H^1}}{\partial t} \lesssim ||u||^2_{H^\alpha} ||u||^2_{H^1} + ||u||^2_{L^2}.$$

In turn, combining this with (3.4.4) and with Gronwall's inequality gives global solvability. ■

CHAPTER 4

DYADIC MODELS AND PARTIAL REGULARITY RESULTS

4.1 Introduction

In this chapter we introduce dyadic models for the equations of fluid motion. We prove partial regularity result for the dyadic Navier-Stokes equations with hyper-dissipation. In the chapters following this chapter we will prove the same result for the actual Navier-Stokes equations with hyper-dissipation. The dyadic model illustrates main ideas used in subsequent chapters in proving an estimate for the Hausdorff dimension of the set of singular points for the Navier-Stokes equation with hyper-dissipation at the first time of blow-up.

Also we prove partial regularity result for the dyadic version of the Ladyzhenkaya's modification of the Navier-Stokes equation. This proof is slightly different from one for the dyadic Navier-Stokes equations with hyper-dissipation and requires a more genuine consideration of energy decay.

The dyadic model presented here is an infinite system of ODEs. Each ODE is given in terms of a wavelet coefficient which describes behavior of the velocity that is localized to a certain frequency range. Therefore the dyadic model could be understood in a general context of Littelwood-Paley theory. Also each ODE reflects behavior of the velocity localized in space on a dyadic cube. For such an ODE we figure out scaling balance between the nonlinear term and the dissipation term. When the dissipation term dominates the nonlinear term we are in a subcritical situation, and the growth of a solution to the ODE is controlled. That is all we need, because in the opposite situation, when the nonlinear term dominates the dissipation term we use a lemma which gives an upper bound for the Hausdorff dimension of a "bad" set, provided that we are able to discretize that set in a certain way.

Dyadic models introduced in this chapter have the property of conserved energy, or in the case of the Navier-Stokes equations the property of energy decay. On the other hand the dyadic models have the nonlinear term with a built-in dispersive feature. Thus as models they could be useful in studying global properties of the Navier-Stokes equations.

We note that those models go into a general class of shell models introduced by Gledzer (14) and Ohkitani-Yamada (34). For a survey of mathematical developments in connection with shell models see for example Bohr et al (2).

The chapter is organized as follows. In section 4.2 we present preliminaries. In section 4.3 we introduce dyadic models. Then in sections 4.4 and 4.5 we prove partial regularity results for the dyadic Navier-Stokes equations with hyper-dissipation and for the dyadic version of the Ladyzhenskaya's modification of the Navier-Stokes equations, respectively.

4.2 Preliminaries

Here we will recall the definition of Hausdorff dimension and present a lemma which can be used as a tool in the process of proving an upper bound on the Hausdorff dimension.

Given any set $A \subset \mathbb{R}^n$, the d-dimensional Hausdorff measure of A, $\mathcal{H}^d(A)$, is given by:

$$\mathcal{H}^d(A) = \lim_{\rho \longrightarrow 0} \mathcal{C}^d_\rho(A),$$

where $\mathcal{C}^d_\rho(A)$ is defined as:

$$\mathcal{C}^d_\rho(A) = \inf_{C \in \mathcal{C}_\rho(A)} \sum_{B \in C} r(B)^d.$$

Here $r(B)$ is the radius of ball B and $\mathcal{C}_\rho(A)$ stands for the set of all coverings of A by balls of radius less than or equal to ρ.

Having defined Hausdorff measure we can speak about the Hausdorff dimension which is given by:

$$\inf_{\mathcal{H}^d(A)=0} d.$$

Our goal is to be able to find an upper bound on the Hausdorff dimension of a certain set. In order to do that we will use Lemma 4.2.1. Intuitively speaking the lemma gives an upper bound for the Hausdorff dimension of a certain set, if we are able to discretize the particular set so that at level j it could be seen as collections of not more than 2^{jd} balls of radius 2^{-j}. More precisely:

Lemma 4.2.1 *Let $A_1, \ldots, A_j, \ldots$ be a sequence of collections of balls in $\mathbb{R}^n$ so that each element of A_j has radius 2^{-j}. Suppose that $\#(A_j) \leq 2^{jd}$. Define*

$$A = \limsup_{j \longrightarrow \infty} A_j,$$

to be the set of points in infinitely many of the $\cup_{B \in A_j} B$'s. Then the Hausdorff dimension of A is at most d.

Proof From the definition of the Hausdorff dimension of A we see that is suffices to prove:

$$\mathcal{H}^{\gamma}(A) = 0, \text{ for all } \gamma > d.$$

Choose j such that $2^{-j} < \rho$. Then A can be covered by the $\cup_{k>j} \cup_{B \in A_k} B$.

Thus

$$\mathcal{H}^{\gamma}(A) \leq \sum_{k>j} 2^{kd}(2^{-k})^{\gamma},$$

and the limit as j goes to ∞ of the right hand side is zero whenever $\gamma > d$. ∎

4.3 Dyadic model

Here we shall introduce a dyadic model for the equations of fluid motion in three dimensions.

We define a dyadic cube in a standard way. A cube Q in $\mathbb{R}^3$ is a dyadic cube if its sidelength is an integer power of 2, 2^l, and the corners of the cube are on the lattice $2^l\mathbb{Z}^3$.

We let $\mathcal{D}$ denote the set of dyadic cubes in $\mathbb{R}^3$. We let $\mathcal{D}_j$ denote the subset of dyadic cubes having sidelength 2^{-j}. Abusing notation slightly we define the function

$$j : \mathcal{D} \longrightarrow \mathbb{Z},$$

by letting $j(Q) = j$ if $Q \in \mathcal{D}_j$. We define $\tilde{Q}$, the parent of Q, to be the unique dyadic cube in $\mathcal{D}_{j(Q)-1}$ which contains Q. For $m \geq 1$ we define $\mathcal{C}^m(Q)$, the mth order grandchildren of Q to be the set of those cubes in $\mathcal{D}_{j(Q)+m}$ which are contained in Q. We sometimes refer to the first order grandchildren of Q as the children of Q.

In our dyadic model we consider a scalar valued function u. It is represented by a wavelet expansion:

$$u = \sum_Q u_Q w_Q,$$

where $\{w_Q\}$ is an orthonormal family of wavelets such that the wavelet w_Q is associated to the spatial dyadic cube $Q \in \mathcal{D}_j$. The wavelet coefficient corresponding to the cube Q is denoted by u_Q. We will refer to the values u_Q as the coefficients of the function u.

We define the dyadic Laplacian Δ by:

$$\Delta(w_Q) = 2^{2j} w_Q.$$

We define $||u||_{L^2}$ to denote the L^2 norm of u and for any $\alpha > 0$, we define

$$||u||_{H^{2\alpha}} = ||u||_{L^2} + ||(\Delta)^\alpha u||_{L^2}.$$

We define $\langle u, v\rangle$ to denote the L^2 pairing of u and v.

We would like to have an operator which will mimic the behavior of the nonlinear term $u \cdot \nabla u$. Note that

$$||w_Q||_{L^\infty} \sim 2^{\frac{3j(Q)}{2}}. \tag{4.3.1}$$

On the other hand

$$||\nabla w_Q||_{L^2} \sim 2^j, \tag{4.3.2}$$

since our wavelets w_Q's are orthonormal and localized to the frequencies around 2^j.

With respect to (4.3.1) and (4.3.2) we define a bilinear operator, the cascade operator, by defining two pieces from which it is built. The cascade down operator is defined by

$$(C_d(u,v))_Q = 2^{\frac{5j(Q)}{2}} u_{\tilde{Q}} v_{\tilde{Q}}.$$

We define the cascade up operator by

$$(C_u(u,v))_Q = 2^{\frac{5(j(Q)+1)}{2}} u_Q \sum_{Q' \in C^1(Q)} v_{Q'}.$$

We define the cascade operator

$$C(u,v) = C_u(u,v) - C_d(u,v).$$

Obviously,

$$\langle C_u(u,u), u\rangle = \langle C_d(u,u), u\rangle,$$

which implies

$$\langle C(u,u), u\rangle = 0. \tag{4.3.3}$$

Having defined operator $C(u,v)$ we can speak about dyadic version of the Euler as well as the Navier-Stokes equations. More precisely by the dyadic Euler equations we mean:

$$\frac{du}{dt} + C(u,u) = 0.$$

Now we introduce the dyadic Navier-Stokes equations as:

$$\frac{du}{dt} + C(u,u) + \Delta u = 0.$$

Also we could speak about the dyadic Navier-Stokes equations with hyper-dissipation by which we mean:

$$\frac{du}{dt} + C(u,u) + (\Delta)^{\alpha} u = 0. \tag{4.3.4}$$

In a similar fashion we define a dyadic model for the Ladyzhenskaya's modification of the Navier-Stokes equations:

$$\frac{du}{dt} + C(u,u) - \operatorname{div}\, T(D) = 0, \tag{4.3.5}$$

where

$$D = \nabla u,$$

and the stress tensor T satisfies conditions:

$$(i)\ |T_{ik}(D)| \leq c_1(1 + |D|^{2\mu})|D|, \tag{4.3.6}$$

$$(ii)\ T_{ik}(D)(\frac{\partial u_i}{\partial x_k}) \geq \nu_0 D^2 + \nu_1 D^{2+2\mu}. \tag{4.3.7}$$

A simple consequence of (4.3.3) is conservation of energy for all four equations, which is an important feature of the actual equations preserved in the dyadic model.

4.4 Partial regularity for the dyadic Navier-Stokes equations with hyper-dissipation

In this section we will investigate partial regularity results for the equation (4.3.4).

We consider the dyadic Navier-Stokes equation with hyper-dissipation (4.3.4). We would like to estimate Hausdorff dimension of the set of singular points at the first time of blow up, T. The nonlinear term $C(u, u)$ on scale j looks like $2^{\frac{5j}{2}} u_Q^2$ (if we imagine for a moment that all neighboring cubes have coefficients of roughly the same size), while the dissipation term gives decay like $2^{2\alpha j} u_Q$. This means that as long as

$$|u_Q| < 2^{-\frac{j}{2}(5-4\alpha)},$$

the growth of u_Q is under control. We shall call this bound on the coefficients "critical regularity". But let us check what happens if $|u_Q| > 2^{-\frac{j}{2}(5-4\alpha)}$.

We can rewrite equation (4.3.4) in terms of wavelets coefficients as follows:

$$\frac{du_Q}{dt} = \sum_{Q',Q''\in\mathcal{E}(Q)} c_{(Q,Q',Q'')} 2^{\frac{5j(Q)}{2}} u_{Q'} u_{Q''} - 2^{2\alpha j(Q)} u_Q, \tag{4.4.1}$$

where $\mathcal{E}(Q) := \{\tilde{Q}, Q\} \cup \mathcal{C}^1(Q)$, and

$$c_{(Q,Q',Q'')} = \left\{ \begin{array}{ll} 1, & \text{if } Q' = Q'' = \tilde{Q} \\ -2^{\frac{5}{2}}, & Q' = Q \text{ and } Q'' \in \mathcal{C}^1(Q) \\ 0, & \text{otherwise} \end{array} \right\}.$$

Having assumed $|u_Q| \gtrsim 2^{-\frac{j}{2}(5-4\alpha)}$ for some time t and assuming that at the initial time $t = 0$ it is much smaller, by the smoothness assumption on the initial condition, we integrate (4.4.1) in time on the interval $[0, T]$ and obtain for one of the choices of (Q', Q'') giving a non-vanishing coefficient:

$$2^{\frac{5j}{2}} \int_0^T |u_{Q'} u_{Q''}| dt \gtrsim 2^{-\frac{j}{2}(5-4\alpha)},$$

which by Cauchy-Schwartz implies:

$$2^{\frac{5j}{2}} (\int_0^T u_{Q'}^2 dt)^{\frac{1}{2}} (\int_0^T u_{Q''}^2 dt)^{\frac{1}{2}} \gtrsim 2^{-\frac{j}{2}(5-4\alpha)}.$$

The last expression can be rewritten as:

$$2^{2j\alpha} (\int_0^T u_{Q'}^2 dt)^{\frac{1}{2}} (\int_0^T u_{Q''}^2 dt)^{\frac{1}{2}} \gtrsim 2^{-j(5-4\alpha)}.$$

This can happen if either

$$2^{2j\alpha}\int_0^T u_{Q'}^2 dt \gtrsim 2^{-j(5-4\alpha)}, \tag{4.4.2}$$

or

$$2^{2j\alpha}\int_0^T u_{Q''}^2 dt \gtrsim 2^{-j(5-4\alpha)}. \tag{4.4.3}$$

However having in mind conservation of energy we have

$$2^{2j\alpha}\int_0^T \sum_{Q\ at\ scale\ j} u_Q^2 dt \lesssim 1.$$

Thus we conclude that (4.4.2) or (4.4.3) could happen in at most $\lesssim 2^{j(5-4\alpha)}$ cubes Q. Now we invoke Lemma 4.2.1 and conclude that the Hausdorff dimension of the set of points of the equation (4.3.4) at which critical regularity fails is at most $5-4\alpha$.

We still need to prove regularity on the interior of a dyadic cube Q, provided that one has a little better than critical regularity at a cube Q. Before formulating a lemma which proves such a statement let us define the graph distance as follows:

Definition 4.4.1 *Let Q be a dyadic cube and let Q_1 be a cube in $\mathcal{C}^m(Q)$. We define $d(Q_1)$, the graph distance of Q_1 to the boundary of Q by $d(Q_1) = m$.*

Then we prove the statement:

Lemma 4.4.2 *Fix an $\epsilon > 0$. Let Q be a dyadic cube in $\mathcal{D}_j$. Suppose we know that for all $t < T$ we have*

$$|u_Q(t)| \lesssim 2^{-\frac{j}{2}(5-4\alpha)},$$

then for any dyadic cube $Q_1 \subset Q$ of length 2^{-k}, we have the estimate

$$|u_{Q_1}(t)| \lesssim 2^{-k\rho(Q_1)}, \tag{4.4.4}$$

where

$$\rho(Q_1) = \frac{5-4\alpha}{2} + \frac{\epsilon d(Q_1)}{2}.$$

Proof The proof we present is by contradiction.

Notice that for all cubes Q_1 of sidelength 2^{-k} we have

$$|u_{Q_1}(0)| \lesssim 2^{-1000k}, \tag{4.4.5}$$

i.e. the lemma is satisfied at time $t = 0$.

Now let t_1 be the first time at which lemma fails, and let Q_1 be one of the cubes at which the lemma fails. Thus

$$|u_{Q_1}(t_1)| \gtrsim 2^{-k\rho(Q_1)}. \tag{4.4.6}$$

Having in mind (4.4.5), we can find the time t_0, being the last time before t_1 when

$$|u_{Q_1}(t_0)| \lesssim 2^{k(-\rho(Q_1)+\epsilon)}. \tag{4.4.7}$$

Thus

$$u^2_{Q_1}(t) \gtrsim 2^{k(-2\rho(Q_1)+2\epsilon)}, \text{ for all time } t \text{ such that } t_0 < t < t_1. \tag{4.4.8}$$

But the lemma is satisfied on the time interval (t_0, t_1).

We multiply the equation (4.4.1) by u_{Q_1}. We observe that for any time t between t_0 and t_1, the dissipation term at the cube Q_1 satisfies:

$$2^{2\alpha k} u^2_{Q_1} \gtrsim 2^{(2\alpha-2\rho(Q_1)+2\epsilon)k}. \tag{4.4.9}$$

Therefore to reach contradiction it suffices to prove that for any time t on the interval (t_0, t_1), the nonlinear term cannot reach the dissipation term, i.e.

$$\sum_{Q_1', Q_1'' \in \mathcal{E}(Q_1)} c_{(Q_1,Q_1',Q_1'')} 2^{\frac{5k(Q_1)}{2}} u_{Q_1} u_{Q_1'} u_{Q_1''} \lesssim 2^{(2\alpha-2\rho(Q_1)+2\epsilon)k}, \tag{4.4.10}$$

where

$$c_{(Q_1,Q_1',Q_1'')} = \left\{ \begin{array}{ll} 1, & \text{if } Q_1' = Q_1'' = \tilde{Q}_1 \\ -2^{\frac{5}{2}}, & Q_1' = Q_1 \text{ and } Q_1'' \in \mathcal{C}^1(Q_1) \\ 0, & \text{otherwise} \end{array} \right\}.$$

In the rest of the proof we shall use the advantage of the fact that the lemma is satisfied on the time interval (t_0, t_1). First notice that by (4.4.4) we have $|u_{Q_1}| < 2^{-k\rho(Q_1)}$, for all $t \in (t_0, t_1)$.

Now for any $Q_2 \in \mathcal{E}(Q_1)$ we have

$$d(Q_2) \geq d(Q_1) - 1. \tag{4.4.11}$$

This follows from the definition of $\mathcal{E}(Q_1)$.

Using (4.4.4) and (4.4.11), we observe that for any Q_2 in $\mathcal{E}(Q_1)$ such that $Q_2 \subset Q$ and for all $t \in (t_0, t_1)$ we have

$$\begin{aligned} u_{Q_2} &\lesssim 2^{-k\rho(Q_2)} \\ &= 2^{-k(\frac{5-4\alpha}{2} + \frac{\epsilon d(Q_2)}{2})} \\ &\leq 2^{-k(\frac{5-4\alpha}{2} + \frac{\epsilon(d(Q_1)-1)}{2})} \\ &= 2^{-k\rho(Q_1) + \frac{k\epsilon}{2}}, \end{aligned}$$

and therefore

$$u_{Q_2} \lesssim 2^{(-\rho(Q_1) + \frac{\epsilon}{2})k}. \tag{4.4.12}$$

Here we remark that in special case when Q_2, an element of $\mathcal{E}(Q_1)$, coincides with the cube Q itself, we can still obtain (4.4.12). In this case it must be that $Q_1 \in \mathcal{C}^1(Q)$, and therefore the sidelength of Q_1 is $k := j+1$, and $d(Q_1) = 1$. Thus by using the assumption of the lemma we obtain

$$\begin{aligned} u_{Q_2} &\lesssim 2^{-j(\frac{5-4\alpha}{2})} \\ &= 2^{-k(\frac{5-4\alpha}{2})+(k-j)\frac{5-4\alpha}{2}} \\ &\approx 2^{-k(\frac{5-4\alpha}{2})} \\ &= 2^{-k(\frac{5-4\alpha}{2}+\frac{\epsilon(d(Q_1))}{2})+\frac{k\epsilon}{2}} \\ &= 2^{-k\rho(Q_1)+\frac{k\epsilon}{2}}. \end{aligned}$$

On the other hand for any Q_2 in $\mathcal{E}(Q_1)$ we have

$$u_{Q_2} \lesssim 2^{\frac{(4\alpha-5)k}{2}}, \tag{4.4.13}$$

by the lower bound on ρ.

Thus by using (4.4.12) and (4.4.13) we bound the nonlinear term in the following way:

$$\sum_{Q_1', Q_1'' \in \mathcal{E}(Q_1)} c_{(Q_1, Q_1', Q_1'')} 2^{\frac{5k(Q_1)}{2}} u_{Q_1} u_{Q_1'} u_{Q_1''} \lesssim 2^{(2\alpha+\frac{\epsilon}{2}-2\rho(Q_1))k}. \tag{4.4.14}$$

Therefore the nonlinear term cannot contribute to the growth of u_{Q_1}, and u_{Q_1} could not have grown which is a contradiction. ∎

4.5 Partial regularity for a dyadic version of the Ladyzhenskaya's modification of the Navier-Stokes equations

Now let us present partial regularity results for a dyadic model of type (4.3.5). Before introducing our model equation, let us consider the equation (4.3.5) where the stress tensor T is given by:

$$T(D) = \nu |D|^{2\mu} D. \tag{4.5.1}$$

In order to analyze the dissipation term div $T(D)$ on scale j, we imagine that all neighboring cubes have wavelet coefficients of approximately the same size. Since L^∞ is an algebra, and our wavelets w_Q's are orthonormal and localized to the frequencies around 2^j, by recalling (4.3.1) we have:

$$|| \, |\nabla w_Q|^{2\mu} ||_{L^\infty} \lesssim (2^{\frac{3j(Q)}{2}} \cdot 2^j)^{2\mu}, \tag{4.5.2}$$

and

$$||\nabla w_Q||_{L^2} \sim 2^j. \tag{4.5.3}$$

Thus the maximum dissipation on scale j arising from a dyadic model of the type (4.3.5) with the stress tensor (4.5.1) gives decay like $2^{j(5\mu+2)} |u_Q|^{2\mu} u_Q$.

Now we construct the dyadic model for the modified Navier-Stokes equations with dissipative term of the order $2^{j(5\mu+2)}|u_Q|^{2\mu}u_Q$ at scale j. More precisely, we consider the following equation:

$$\frac{du}{dt} + c(u,u) - \text{div } \tilde{T}(D) = 0, \tag{4.5.4}$$

where

$$D = \nabla u,$$

and the stress tensor $\tilde{T}$ is such that $\langle \text{div } \tilde{T}(D), w_Q \rangle = -2^{j(5\mu+2)}|u_Q|^{2\mu}u_Q$, for all dyadic cubes $Q \in \mathcal{D}_j$.

We would like to estimate the Hausdorff dimension of the set of singular points for (4.5.4) at the time τ_1 of first blow up. In order to introduce a certain critical level of regularity we will imagine that all neighboring cubes have wavelet coefficients of approximately the same size. Then the nonlinear term $c(u,u)$ on scale j looks like $2^{\frac{5j}{2}}u_Q^2$, while the dissipation term on scale j gives decay like $2^{j(5\mu+2)}|u_Q|^{2\mu}u_Q$. We note that in terms of scaling the nonlinear term is controlled by the dissipation term on level j as long as:

$$2^{j(5\mu+2)}|u_Q|^{2\mu+1} > 2^{\frac{5j}{2}}u_Q^2. \tag{4.5.5}$$

In order to simplify our notation we denote $\frac{j(1-10\mu)}{2(1-2\mu)}$ by $r(j)$. Thus for $0 \le \mu < \frac{1}{2}$ we can rewrite (4.5.5) as:

$$|u_Q| < 2^{-r(j)}.$$

We call this level of regularity "critical regularity".

Here we will investigate what happens if

$$|u_Q| \gtrsim 2^{-r(j)}, \text{ at the first time } \tau_1. \tag{4.5.6}$$

Thus τ_1 is introduced as

$$\tau_1 = \inf\{\tau > 0 : |u_Q(\tau)| > 2^{-r}\}.$$

Let τ_0 be defined as:

$$\tau_0 = \sup\{\tau \in [0, \tau_1) : |u_Q(\tau)| < c2^{-r}, \text{ with } c < 1\}.$$

Such time τ_0 exists, because otherwise we would contradict initial condition on $u_Q(0)$. Thus we have:

$$|u_Q(t)| \gtrsim 2^{-r}, \text{ for all time } t \in (\tau_0, \tau_1). \tag{4.5.7}$$

We can rewrite equation given with (4.5.4) in terms of wavelets coefficients as an infinite system of ODEs:

$$\frac{du_Q}{dt} = \sum_{Q', Q'' \in \mathcal{E}(Q)} c_{(Q,Q',Q'')} 2^{\frac{5j(Q)}{2}} u_{Q'} u_{Q''} - 2^{j(5\mu+2)} |u_Q|^{2\mu} u_Q, \tag{4.5.8}$$

where

$$c_{(Q,Q',Q'')} = \left\{ \begin{array}{ll} 1, & \text{if } Q' = Q'' = \tilde{Q} \\ -2^{\frac{5}{2}}, & Q' = Q \text{ and } Q'' \in \mathcal{C}^1(Q) \\ 0, & \text{otherwise} \end{array} \right\}.$$

A simple consequence of (4.3.3) is conservation of energy for (4.5.8),

$$2^{j(5\mu+2)} \sum_{Q \; at \; scale \; j} \int_0^{\tau_1} |u_Q|^{2\mu} u_Q^2 dt \lesssim 1. \tag{4.5.9}$$

which is an important feature of this equation that we will use here.

By using our assumption (4.5.7) we have

$$|u_Q|^{2\mu} u_Q^2 \gtrsim 2^{-2\mu r} u_Q^2, \text{ for all time } t, \; \tau_0 < t < \tau_1, \tag{4.5.10}$$

and therefore

$$2^{-2\mu r} \int_{\tau_0}^{\tau_1} u_Q^2 dt \lesssim \int_{\tau_0}^{\tau_1} |u_Q|^{2\mu} u_Q^2 dt \lesssim \int_0^{\tau_1} |u_Q|^{2\mu} u_Q^2 dt. \tag{4.5.11}$$

Hence the energy integral (4.5.9) implies:

$$2^{j(5\mu+2)} 2^{-2\mu r} \sum_{Q \; at \; scale \; j} \int_{\tau_0}^{\tau_1} u_Q^2 dt \lesssim 1. \tag{4.5.12}$$

Having assumed $|u_Q| \gtrsim 2^{-r}$ for time t, $\tau_0 < t < \tau_1$, we integrate (4.5.8) in time on the interval $[\tau_0, \tau_1]$. By recalling the meaning of the symbols $\lesssim$ and $\gtrsim$ and by noticing that

$$|2^{j(5\mu+2)} \int_{\tau_0}^{\tau_1} |u_Q|^{2\mu} u_Q| \lesssim | \int_{\tau_0}^{\tau_1} \sum_{(Q',Q'')} c_{(Q,Q',Q'')} 2^{\frac{5j(Q)}{2}} u_{Q'} u_{Q''}|, \tag{4.5.13}$$

we obtain for one of the choices of (Q', Q'') giving a non-vanishing coefficient:

$$2^{\frac{5j}{2}} \int_{\tau_0}^{\tau_1} |u_{Q'} u_{Q''}| dt \gtrsim 2^{-r}, \tag{4.5.14}$$

which by Cauchy-Schwartz implies:

$$2^{\frac{5j}{2}} (\int_{\tau_0}^{\tau_1} u_{Q'}^2 dt)^{\frac{1}{2}} (\int_{\tau_0}^{\tau_1} u_{Q''}^2 dt)^{\frac{1}{2}} \gtrsim 2^{-r}. \tag{4.5.15}$$

After we multiply both sides of (4.5.15) with $2^{j(5\mu+2)} 2^{-2\mu r}$ it becomes:

$$2^{j(5\mu+2)} 2^{-2\mu r} (\int_{\tau_0}^{\tau_1} u_{Q'}^2 dt)^{\frac{1}{2}} (\int_{\tau_0}^{\tau_1} u_{Q''}^2 dt)^{\frac{1}{2}} \gtrsim 2^{-r - \frac{5j}{2} + j(5\mu+2) - 2\mu r}. \tag{4.5.16}$$

With respect to the definition of r, (4.5.16) becomes

$$2^{j(5\mu+2)} 2^{-2\mu r} (\int_{\tau_0}^{\tau_1} u_{Q'}^2 dt)^{\frac{1}{2}} (\int_{\tau_0}^{\tau_1} u_{Q''}^2 dt)^{\frac{1}{2}} \gtrsim 2^{-j \frac{1-10\mu}{1-2\mu}}. \tag{4.5.17}$$

This can happen if either

$$2^{j(5\mu+2)}2^{-2\mu r}\int_{\tau_0}^{\tau_1} u_{Q'}^2\,dt \gtrsim 2^{-j\frac{1-10\mu}{1-2\mu}}, \tag{4.5.18}$$

or

$$2^{j(5\mu+2)}2^{-2\mu r}\int_{\tau_0}^{\tau_1} u_{Q''}^2\,dt \gtrsim 2^{-j\frac{1-10\mu}{1-2\mu}}. \tag{4.5.19}$$

Thus by recalling (4.5.12) we conclude that (4.5.18) or (4.5.19) could happen in at most $\lesssim 2^{j\frac{1-10\mu}{1-2\mu}}$ cubes Q. Now we invoke Lemma 4.2.1 and conclude that the Hausdorff dimension of the set of singular points of (4.3.5) at which critical regularity fails is at most $\frac{1-10\mu}{1-2\mu}$.

In order to complete the dyadic heuristic we still need to prove regularity on the interior of a dyadic cube Q, provided that one has critical regularity for cubes containing it. This could be done in a similar way as in the case of the equation (4.3.4) and we omit the proof.

CHAPTER 5

LOCALIZED NAVIER-STOKES EQUATIONS WITH HYPER-DISSIPATION

5.1 Introduction

In Chapter 4 we presented the dyadic heuristic which gave an upper bound on the Hausdorff dimension of the singular set for the dyadic Navier-Stokes equations with hyper-dissipation. Now we would like to generalize such a result for the actual Navier-Stokes equation with hyper-dissipation:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = -(-\Delta)^{\alpha} u, \tag{5.1.1}$$

where u is a time-dependent divergence free vector field in $\mathbb{R}^3$. One sets the initial condition

$$u(x, 0) = u_0(x) \tag{5.1.2}$$

where $u_0(x) \in C_c^{\infty}(\mathbb{R}^3)$.

First we need to localize the equation (5.1.1) in frequency and space. We do that in this chapter. In particular to deal with localization we combine the theory of paramultiplication presented in chapter 2 with the theory of pseudodifferential operators. Once equation (5.1.1) is in a localized form we analyze the nonlinear term and the dissipation term separately. We obtain an upper bound for the nonlinear term, and a lower bound for the dissipation term. Then we figure out a balance between these two terms. For each localization scale we introduce a bad cube as a cube where the nonlinear term dominates the dissipation term. However by using covering lemma of Vitali we are able to count number of bad cubes at each scale, which is enough to obtain an estimate on the Hausdorff dimension of the singular set. We do all that in this chapter. Then in chapters 6 and 7 we prove regularity outside of the bad set.

This chapter is organized as follows. In section 5.2 we employ Littlewood-Paley operators and pseudodifferential operators to localize the equation (5.1.1) in frequency and space. In section 5.3 we analyze the dissipation term. In section 5.4 we obtain an upper bound on the nonlinear term. In section 5.5 we describe the singular set and its covering.

5.2 Littlewood Paley theory and Pseudodifferential operators

Here we localize the equation (5.1.1) in frequency and space.

We shall use a standard Littlewood Paley partition of frequency space as introduced in chapter 1. The idea of Littlewood-Paley theory is that $P_j f$ is like a combination of wavelets supported on cubes of length 2^{-j}. Bernstein's inequality is sharp only when a large proportion of the L^2 energy of $P_j f$ is concentrated in a single one of these cubes. Hence one is led to try to localize $P_j f$ in space on such cubes. However by Heisenberg uncertainty principle we cannot achieve perfect localization both in frequency and space. It is important to notice that we have some room for error since our goal is a Hausdorff dimension estimate which is a closed condition. Therefore we fix an $\epsilon > 0$ and never try to localize better than within $2^{-j(1-\epsilon)}$.

Our localization is nearly perfect if we ignore negligible quantities. More precisely, whenever we are at scale j, we neglect quantities of size $\lesssim 2^{-100j}$ since they will not affect our estimates. Similarly we neglect operators whose norms are smaller than 2^{-100j} provided they will only be applied to functions whose norms are $\lesssim 1$. The choice of 100 is arbitrary. It is large enough so that it does not affect the L^∞ norm of u. But in fact our techniques for showing quantities are negligible rely on Schwartz function properties and could give an arbitrary exponent with loss in the constant. The current approach works well principally because of conservation of energy, since the L^2 norm of u is certainly $\lesssim 1$.

Now let us localize in space. For any cube Q with sidelength greater than $2^{-j(1-2\epsilon)}$, we define a bump function $\phi_{Q,j}$ which is positive, is bounded above by 1, equals 1 on Q and is 0 outside of $(1+2^{-j\epsilon})Q$. Further, we require for each multiindex α that there is a constant C_α independent of Q so that

$$|D^\alpha \phi_{Q,j}| \leq C_\alpha 2^{|\alpha| j(1-\epsilon)}. \tag{5.2.1}$$

We say that any bump function which satisfies estimates (5.2.1) is of type j.

The map $\phi_{Q,j}P_j$ acts much like a projection, and we shall treat $||\phi_{Q,j}P_j f||_{L^2}$ as if it were a wavelet coefficient. When we deal with a cube Q of sidelength $2^{-j(1-2\epsilon)}$, we shall define $j(Q)=j$ and we shall denote $||\phi_{Q,j(Q)}P_j f||_{L^2}$ by f_Q. Further if $j(Q)=j$, we say that Q is at level j.

Here we note that $\phi_{Q,j}P_j$ is a pseudodifferential operator. In order to see that we recall definition of a pseudodifferential operator according to Taylor (47) .

Definition 5.2.1 *We say that the operator*

$$p(x,D)f(x) := \int p(x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}d\xi$$

with a symbol $p(x,\xi)$ is in class $OPS^m_{\rho,\delta}$ if there is a constant $C_{a,b}$ such that:

$$|D^b_x D^a_\xi p(x,\xi)| \leq C_{a,b} \langle \xi \rangle^{m-\rho|a|+\delta|b|},$$

for all a, b, where $\langle \xi \rangle = (1+|\xi|^2)^{\frac{1}{2}}$, and $\rho, \delta \in [0,1]$. Here m stands for the order of the pseudodifferential operator, and (ρ,δ) for its type.

Thus $\phi_{Q,j}P_j$, with the symbol $\phi_{Q,j}(x)p_j(\xi)$, is a pseudodifferential operator in class $OPS^0_{1,1-\epsilon}$. Now we will show that $\phi_{Q,j}P_j$ is localized in frequency, up to negligible terms.

Proposition 5.2.2 *Given f with $||f||_{L^2} \lesssim 1$, and ϕ a bump function of type j, the quantity*

$$||\phi P_j f - \tilde{P}_j \phi P_j f||_{L^2}$$

is negligible.

Proof Let us define $\phi = \phi_1 + \phi_2$, where $\hat{\phi}_2(\xi) = \chi_{\{|\xi| > \frac{1}{100}2^j\}} \hat{\phi}(\xi)$. We have

$$\phi P_j f - \tilde{P}_j \phi P_j f = \phi_1 P_j f - \tilde{P}_j \phi_1 P_j f + \phi_2 P_j f - \tilde{P}_j \phi_2 P_j f.$$

By our estimates on the derivatives of ϕ, we can get

$$||\phi_2(\xi)||_{L^\infty} = \text{negligible},$$

while, because of the Fourier transform supports,

$$\tilde{P}_j \phi_1 P_j f = \phi_1 P_j f.$$

Thus the proposition is proved. ■

We would like to have a Bernstein's type inequality for $\phi_{Q,j} P_j$ too, i.e. an inequality which allows us to move from $||\phi_{Q,j} P_j||_{L^2}$ to $||\phi_{Q,j} P_j||_{L^\infty}$. Combining Bernstein's inequality itself and Proposition 5.2.2, we get the following extremely useful lemma, which we could think of as "super Bernstein's inequality".

Lemma 5.2.3 *Let ϕ be a bump function of type j. Then*

$$||\phi P_j f||_{L^\infty} \lesssim 2^{\frac{3j}{2}} ||\phi P_j f||_{L^2} + \text{negligible}.$$

Proof We neglect negligible terms. Then we have $\phi P_j f = \tilde{P}_j \phi P_j f$, by proposition 5.2.2. We estimate $||\tilde{P}_j \phi P_j f||_{L^\infty}$ by Bernstein's inequality. ■

Before formulating the proposition which shows that $\phi_{Q,j} P_j$ is localized in space let us recall asymptotic formula for product of two pseudodifferential operators that will be useful in the proof of Proposition 5.2.4. We state the product rule according to (47). More precisely, if $p_j(x,\xi) \in OPS^{m_j}_{\rho_j,\delta_j}$, and $0 \leq \delta_2 < \rho \leq 1$ with $\rho = \min(\rho_1, \rho_2)$, then

$$p_1(x,D) p_2(x,D) = q(x,D) \in OPS^{m_1+m_2}_{\rho,\delta},$$

with $\delta = \max(\delta_1, \delta_2)$, and

$$q(x,\xi) \sim \sum_{\beta \geq 0} \frac{i^{|\beta|}}{\beta!} D_\xi^\beta p_1(x,\xi) D_x^\beta p_2(x,\xi). \tag{5.2.2}$$

Now we show that $\phi_{Q,j} P_j$ is localized in space too, up to negligible terms.

Proposition 5.2.4 *For any cube Q, we have that for any f with $||f||_{H^\beta} \lesssim 1$ with $\beta < 10$ that*

$$(1 - \phi_{(1+2^{\frac{-\epsilon j(Q)}{2}})Q,j(Q)}) P_j \phi_{Q,j(Q)} f,$$

is negligible.

Proof

$$(1 - \phi_{(1+2^{\frac{-\epsilon j(Q)}{2}})Q,j(Q)}) P_j \phi_{Q,j(Q)},$$

is a composition of type $(1, 1-\epsilon)$ pseudodifferential operators whose symbols have disjoint support. Thus when we apply formula (5.2.2) the term of order zero is equal to zero, and we are left with higher order derivatives, which is smoothing. ■

Notice that the above proposition really says that we can move bump functions across Littlewood Paley projections as long as the bump functions proliferate and increase in support. In other words, the proposition can be rewritten as

$$\phi_{(1+2^{\frac{-\epsilon j(Q)}{2}})Q,j(Q)} P_j \phi_{Q,j(Q)} = P_j \phi_{Q,j(Q)} + \text{negligible}.$$

Similarly, this can be used to remove inconvenient bump functions, provided there is a smaller bump function already present in the expression.

Proposition 5.2.5 *Let $\mathcal{C}$ be a covering of a set E by cubes of sidelength $2^{-j(1-2\epsilon)}$ for some fixed j. Then for any $f \in L^2$, we have*

$$||\chi_E P_j f||_{L^2}^2 \leq \sum_{Q \in \mathcal{C}} f_Q^2.$$

Proof We simply observe that

$$\sum_{Q \in \mathcal{C}} f_Q^2 = \int (\sum_{Q \in \mathcal{C}} \phi_{Q,j}^2)|P_j f|^2 \geq \int \chi_E |P_j f|^2.$$

■

Essentially what we have done up to now is to use approximations to projections which are uniformly pseudodifferential operators of type $(1, 1-\epsilon)$. The negligible interaction of distant squares can be just as well derived from the asymptotic formula for composition of such operators. The composition of two $(1, 1-\epsilon)$ operators whose symbols have disjoint support is infinitely smoothing and hence negligible.

Now we are ready to write a localized form of the Navier-Stokes equations with hyper-dissipation. Let u be a solution to the equation (5.1.1).

In light of divergence-free condition $\nabla \cdot u = 0$, we can rewrite (5.1.1) as

$$\frac{\partial u}{\partial t} + T(u \cdot \nabla u) = -(-\Delta)^{\alpha} u, \tag{5.2.3}$$

where T is the projection into divergence free vector fields. The operator T is a singular integral operator and is also a Fourier multiplier. We pick a cube Q of side length $2^{-j(1-2\epsilon)}$ and compute the L^2 pairing of the equation with $P_j\phi_{Q,j}^2 P_j u$. We obtain the energy estimate

$$\frac{1}{2}\frac{d}{dt}u_Q^2 = \langle -T(u\cdot\nabla u), P_j\phi_{Q,j}^2 P_j u\rangle - \langle(-\Delta)^\alpha u, P_j\phi_{Q,j}^2 P_j u\rangle \tag{5.2.4}$$

We shall estimate the two terms on the right hand side of (5.2.4) separately.

5.3 Dissipation

In this section we analyze the dissipation term

$$\langle(-\Delta)^\alpha u, P_j\phi_{Q,j}^2 P_j u\rangle.$$

As before we use the notation

$$\tilde{P}_j = \sum_{k=-2}^{2} P_{j+k}.$$

This has the advantage that

$$P_j = \tilde{P}_j P_j.$$

We make some definitions. We define for each cube Q, the set $\mathcal{N}^1(Q)$, the nuclear family of Q to be a union of sets A_Q, B_Q, C_Q, D_Q, E_Q, where A_Q, B_Q, C_Q, D_Q, E_Q are covers of $\tilde{Q} = (1+2^{\frac{-\epsilon j}{4}})Q$ by fewer than 1024 cubes each at levels respectively $j-2, j-1, j, j+1$, and $j+2$. We define recursively $\mathcal{N}^l(Q)$ to be the union of all $\mathcal{N}^1(Q')$ for all $Q' \in \mathcal{N}^{l-1}(Q)$. Thus in particular, we have

$$\#(\mathcal{N}^l(Q)) \leq 2^{13l},$$

since $5(2^{10}) < 2^{13}$.

Now we are ready to state and prove the lower bound on the dissipation term:

Proposition 5.3.1 *Let Q be a cube and $j = j(Q)$. Then*

$$\langle (-\Delta)^\alpha u, P_j \phi_{Q,j}^2 P_j u \rangle \geq c 2^{2\alpha j} u_Q^2 - C 2^{(2\alpha - 2\epsilon)j} \sum_{Q' \in \mathcal{N}^1(Q)} u_{Q'}^2 - \text{negligible} \tag{5.3.1}$$

Proof Note that

$$\begin{aligned} \langle (-\Delta)^\alpha u, P_j \phi_{Q,j}^2 P_j u \rangle &= \langle \phi_{Q,j} P_j (-\Delta)^\alpha u, \phi_{Q,j} P_j u \rangle \\ &= \langle (-\Delta)^\alpha \phi_{Q,j} P_j u, \phi_{Q,j} P_j u \rangle + \langle [\phi_{Q,j} P_j, (-\Delta)^\alpha] u, \phi_{Q,j} P_j u \rangle \\ &= X + Y. \end{aligned}$$

Note that

$$\begin{aligned} X &:= \langle(-\Delta)^{\alpha}\phi_{Q,j}P_j u, \phi_{Q,j}P_j u\rangle \\ &= ||(-\Delta)^{\frac{\alpha}{2}}\phi_{Q,j}P_j u||^2_{L^2} \\ &\gtrsim 2^{2\alpha j}||\tilde{P}_j\phi_{Q,j}P_j u||^2_{L^2} \\ &= 2^{2\alpha j}u_Q^2 - \text{negligible}, \end{aligned}$$

where the last equality follows from Proposition 5.2.2.

To estimate Y we shall use (5.2.2). Let $p_1(x,\xi)$ be a symbol of $(-\Delta)^{\alpha}$, let $p_2(x,\xi)$ be a symbol of $\phi_{Q,j}P_j$, and $q(x,\xi)$ be a symbol of $[(-\Delta)^{\alpha}, \phi_{Q,j}P_j]$. Then we have:

$$\begin{aligned} q(x,\xi) &\sim \sum_{\beta\geq 0}\frac{i^{|\beta|}}{\beta!}D_\xi^\beta p_1(x,\xi)D_x^\beta p_2(x,\xi) - \sum_{\beta\geq 0}\frac{i^{|\beta|}}{\beta!}D_\xi^\beta p_2(x,\xi)D_x^\beta p_1(x,\xi) \\ &= [p_1(x,\xi)p_2(x,\xi) - p_2(x,\xi)p_1(x,\xi)] \\ &+ [\frac{1}{2\pi i}D_\xi p_1(x,\xi)D_x p_2(x,\xi) - \frac{1}{2\pi i}D_\xi p_2(x,\xi)D_x p_1(x,\xi)] \\ &+ [(\frac{1}{2\pi i})^2 D_\xi^2 p_1(x,\xi)D_x^2 p_2(x,\xi) - (\frac{1}{2\pi i})^2 D_\xi^2 p_2(x,\xi)D_x^2 p_1(x,\xi)]... \end{aligned}$$

Since $\phi_{Q,j}P_j$ is a pseudodifferential operator in the class $OPS^0_{1,1-\epsilon}$, and $(-\Delta)^{\alpha}$ is a pseudodifferential operator in the class $OPS^{2\alpha}_{1,0}$, the asymptotic formula for $q(x,\xi)$ implies:

$$|q(x,\xi)| \leq \frac{1}{2\pi i}\langle\xi\rangle^{(2\alpha-\epsilon)} + (\frac{1}{2\pi i})^2\langle\xi\rangle^{(2\alpha-2\epsilon)} + ...$$

Thus we observe that $[\phi_{Q,j}P_j, (-\Delta)^\alpha]$ is of order $2\alpha - \epsilon$. Further by Proposition 5.2.2, we have

$$[\phi_{Q,j}P_j, (-\Delta)^\alpha]\tilde{P}_j - [\phi_{Q,j}P_j, (-\Delta)^\alpha] = \text{negligible}.$$

Even further, applying Proposition 5.2.4, we get

$$Y = \langle [\phi_{Q,j}P_j, (-\Delta)^\alpha]\phi_{\tilde{Q},j}\tilde{P}_j u, \phi_{Q,j}P_j u\rangle + \text{negligible}.$$

Now applying the mapping properties of operators of order $2\alpha - \epsilon$ we get

$$|Y| \leq 2^{(2\alpha-\epsilon)j}||\phi_{\tilde{Q},j}\tilde{P}_j u||_{L^2}||\phi_{Q,j}P_j u||_{L^2} \tag{5.3.2}$$

which for any number K, by Cauchy Schwartz implies

$$|Y| \lesssim \frac{1}{K}2^{2\alpha j}u_Q^2 + K2^{(2\alpha-2\epsilon)j}||\phi_{\tilde{Q},j}\tilde{P}_j u||_{L^2}^2.$$

Now applying Proposition 5.2.5, we get the desired result. ■

Let us define for every l,

$$u_{\mathcal{N}^l(Q)}^2 = \sum_{Q' \in \mathcal{N}^l(Q)} u_{Q'}^2.$$

Then we have

Corollary 5.3.2 *Let Q be a cube and $j = j(Q)$. Then for any l,*

$$\sum_{Q' \in \mathcal{N}^l(Q)} \langle (-\Delta)^\alpha u, P_j \phi^2_{Q',j} P_j u \rangle \geq c 2^{2\alpha j} u^2_{\mathcal{N}^l(Q)} - C 2^{(2\alpha - 2\epsilon)j} u^2_{\mathcal{N}^{l+1}(Q)} - \text{negligible} \tag{5.3.3}$$

Proof Simply sum proposition 5.3.1 over $\mathcal{N}^l(Q)$. ∎

We want to use Proposition 5.3.1 to fight against the growth of u_Q. Thus the term $2^{(2\alpha-2\epsilon)j} u^2_{Q'}$ would appear to be a serious nuisance. However heuristically speaking if the term $2^{(2\alpha-2\epsilon)j} u^2_{Q'}$ is too large compared to the first term on the right-hand side of Proposition 5.3.1 we can extend our consideration from a cube Q' to its nuclear family $\mathcal{N}(Q')$. Then if the term $2^{(2\alpha-2\epsilon)j} u^2_{\mathcal{N}(Q')}$ is still to large we go from $\mathcal{N}(Q')$ to its nuclear family and so on up to $\mathcal{N}^l(Q')$. However we will have to stop at some point because of conservation of energy. We make this precise by the following lemma, which we call "good neighbors".

Lemma 5.3.3 *For any interval of time $J \subset [0, T]$ and any cube Q we may find an $l < \frac{400}{\epsilon}$ so that*

$$\int_J u^2_{\mathcal{N}^l(Q)} dt + \text{negligible} \gtrsim 2^{-\epsilon j} \int_J u^2_{\mathcal{N}^{l+1}(Q)} dt. \tag{5.3.4}$$

Proof By conservation of energy, for every $l \leq \frac{400}{\epsilon}$ and every t we have,

$$u^2_{\mathcal{N}^{l+1}(Q)} \lesssim 1.$$

Since our constants can depend on T this means

$$\int_J u^2_{\mathcal{N}^{l+1}(Q)} dt \lesssim 1.$$

Now suppose the lemma is false. Applying the opposite of (5.3.4), for $l = 1 \dots \frac{400}{\epsilon}$, we get

$$\int_J u^2_{\mathcal{N}^2(Q)} dt + \text{negligible} \lesssim 2^{-399j} = \text{negligible},$$

which implies that (5.3.4) holds for $l = 1$. ■

We will be able to apply Lemma 5.3.3 together with Corollary 5.3.2 to show that for any cube and any interval in time, there is an lth iterated nuclear family for small l which is undergoing dissipation.

5.4 An upper bound on the nonlinear term

Now we turn our attention to the term

$$G_Q = \langle -T(u \cdot \nabla u), P_j \phi^2_{Q,j} P_j u \rangle.$$

We rewrite it as

$$G_Q = \langle -\phi_{Q_j} \tilde{P}_j T P_j (u \cdot \nabla u), \phi_{Q,j} P_j u \rangle.$$

Now we use the "trichotomy". We can write

$$P_j(u \cdot \nabla u) = H_{j,lh} + H_{j,hl} + H_{j,hh} + H_{loc},$$

where the low-high part is given by

$$H_{j,lh} = \sum_{k<j-\frac{1000}{\epsilon}} P_j((P_k u) \cdot \tilde{P}_j \nabla u),$$

the high-low part is given by

$$H_{j,hl} = \sum_{k<j-\frac{1000}{\epsilon}} P_j((\tilde{P}_j u) \cdot P_k \nabla u),$$

the high-high part is given

$$H_{j,hh} = \sum_{k>j+\frac{1000}{\epsilon}} P_j((\tilde{P}_k u) \cdot P_k \nabla u) + \sum_{k>j+\frac{1000}{\epsilon}} P_j((P_k u) \cdot \tilde{P}_k \nabla u),$$

(technically that was the end of the trichotomy) and the local part is given by

$$H_{loc} = \sum_{j-\frac{1000}{\epsilon}<k<j+\frac{1000}{\epsilon}} P_j((\tilde{P}_k u) \cdot P_k \nabla u) + \sum_{j-\frac{1000}{\epsilon}<k<j+\frac{1000}{\epsilon}} P_j((P_k u) \cdot \tilde{P}_k \nabla u).$$

Now we break up G_Q in the obvious way:

$$G_Q = G_{Q,lh} + G_{Q,hl} + G_{Q,hh} + G_{Q,loc},$$

where

$$G_{Q,lh} = \langle -\phi_{Q,j} \tilde{P}_j T H_{j,lh}, \phi_{Q,j} P_j u \rangle,$$

$$G_{Q,hl} = \langle -\phi_{Q,j} \tilde{P}_j T H_{j,hl}, \phi_{Q,j} P_j u \rangle,$$

$$G_{Q,hh} = \langle -\phi_{Q,j}\tilde{P}_j T H_{j,hh}\phi_{Q,j}P_j u\rangle,$$

$$G_{Q,loc} = \langle -\phi_{Q,j}\tilde{P}_j T H_{j,loc}, \phi_{Q,j}P_j u\rangle.$$

Our goal is to find an upper bound on G_Q. Now fixing Q and j we would like to estimate each of $G_{Q,lh}, G_{Q,hl}, G_{Q,hh}$, and $G_{Q,loc}$. Before going into details let us intuitively explain how we obtain these bounds. For example let us heuristically discuss how we obtain a bound on $G_{Q,lh}$. We remark that we are on the level of pieces localized in frequency, i.e. on the level of Littlewood-Paley operators. Then we need to estimate L^2-norm of the product:

$$||P_k u \cdot \tilde{P}_j \nabla u||_{L^2}, \tag{5.4.1}$$

where k stands for a low frequency part, and j for a high frequency part. First we apply the Hölder inequality,

$$||f \cdot g||_{L^2} \leq ||f||_{L^2}||g||_{L^\infty},$$

having in mind that we shall choose a function g to be one of $P_k u$, $\tilde{P}_j \nabla u$ so that we can apply Bernstein's inequality on $||g||_{L^\infty}$ in the most efficient way. Since Bernstein's inequality is more efficient when applied on low-frequency part we choose a function g to be $P_k u$, while we choose $\tilde{P}_j \nabla u$ to be f in Hölder's inequality. All this is on the level of Littlewood-Paley operators only. Indeed we deal with pieces localized in frequency and space too, and instead of $P_k u$ we shall be able to deal with $\phi_l P_k u$. However thanks to the calculus developed in section 5.2, we are able to use the idea just described. In such a way by employing Hölder and "super-Bernstein's" inequalities we obtain an upper bound for the actual G_Q.

We take a moment to be careful about how we localize. For any $k < j$, define $Q_k = 2^{(j-k)(1-2\epsilon)}Q$. For any $k \geq j$ (including j) define $Q_k = (1 + 2^{\frac{-\epsilon k}{2}})Q$.

Then we have

Lemma 5.4.1 *For any $\delta > 0$*

$$|G_{Q,lh}| + |G_{Q,hl}| \lesssim \sum_{k=\delta j}^{j-\frac{1000}{\epsilon}} 2^{\frac{3k}{2}+j} u_{Q_k} u_{\mathcal{N}^1(Q)} u_Q + 2^{j(1+\frac{3\delta}{2})} u_{\mathcal{N}^1(Q)} u_Q + \text{negligible}.$$

Proof We consider for $k < j - \frac{1000}{\epsilon}$, the relevant expression

$$G_{Q,lh,k} = \langle -\phi_{Q,j} P_j T(P_k u \tilde{P}_j \nabla u),\ \phi_{Q,j} P_j u \rangle.$$

We divide this into two cases which are $k < \delta j$ and $k \geq \delta j$. In the second case, we use the idea of Proposition 5.2.4 (we just have to use a higher degree of smoothing than is used in that proposition) to observe

$$G_{Q,lh,k} = \langle -\phi_{Q,j} P_j T((\phi_{Q_k,k} P_k) u \cdot \nabla(\phi_{Q_j,j} \tilde{P}_j u)),\ \phi_{Q,j} P_j u \rangle + \text{negligible}.$$

(This is because ∇ acts on $\phi_{Q_j,j}$ only where it has a negligible effect on the whole quantity by Proposition 5.2.4.) Now we simply observe by Proposition 5.2.3 that

$$||\phi_{Q_k,k} P_k u||_{L^\infty} \lesssim 2^{\frac{3k}{2}} u_{Q_k} + \text{negligible},$$

and by Proposition 5.2.4, the proof of Proposition 5.2.2 (to control the action of ∇) and Proposition 5.2.5 that

$$||\nabla \phi_{Q_j,j} \tilde{P}_j u||_{L^2} \lesssim 2^j u_{\mathcal{N}^1(Q)} + \text{negligible}.$$

On the other hand for the first case, there is no point in localizing to Q_k because k is too small and so the error is too big. Thus in in this case, we simply estimate

$$||P_k u||_{L^\infty} \lesssim 2^{\frac{3\delta j}{2}} u_{Q_k} \lesssim 2^{\frac{3\delta j}{2}}.$$

This last application of conservation of energy accounts for the peculiar homogeneity of our result. We are pleased to control the energy in very large scales by 1.

Summing these estimates gives the desired bound for $G_{Q,lh}$. The bound for $G_{Q,hl}$ proceeds likewise (and gives a better estimate since the derivative falls on the level k term.)

■

Lemma 5.4.2

$$|G_{Q,loc}| \lesssim 2^{\frac{5j}{2}} u_Q u^2_{\mathcal{N}^{\frac{1000}{\epsilon}}(Q)} + \text{negligible}.$$

Proof From the definition of $G_{Q,loc}$ we have:

$$G_{Q,loc} = \sum_{l=-2}^{2} [\sum_{k=j-\frac{1000}{\epsilon}}^{j+\frac{1000}{\epsilon}} \langle \phi_{Q,j} P_j T(P_{k+l} u \cdot \nabla P_k u), \phi_{Q,j} P_j u \rangle + \sum_{k=j-\frac{1000}{\epsilon}}^{j+\frac{1000}{\epsilon}} \langle \phi_{Q,j} P_j T(P_k u \cdot \nabla P_{k+l} u), \phi_{Q,j} P_j u \rangle].$$

Since the above sum has only $\lesssim 1$ many terms by applying Proposition 5.2.4, we observe that for some particular values of k, l

$$|G_{Q,loc}| \lesssim |\langle \phi_{Q,j} P_j T(\phi_{Q_j,j} P_k u \cdot \phi_{Q_j,j} \nabla P_{k+l} u), \phi_{Q,j} P_j u \rangle| + \text{negligible}.$$

Now we apply Proposition 5.2.3 together with Cauchy Schwartz to obtain

$$||\phi_{Q_j,j} P_k u||_{L^\infty} \lesssim 2^{\frac{3j}{2}} ||\phi_{Q_j,j} P_k u||_{L^2} + \text{negligible}.$$

Using Proposition 5.2.5, we observe that

$$||\phi_{Q_j,j} P_k u||_{L^2} \lesssim u_{\mathcal{N}^{\frac{1000}{\epsilon}}(Q)}.$$

Finally direct calculation shows

$$||\phi_{Q_j,j} \nabla P_{k+l} u||_{L^2} \lesssim ||\nabla \phi_{(1+2^{\frac{-2\epsilon j}{3}})Q_j,j} P_{k+l} u|| \lesssim 2^j u_{\mathcal{N}^{\frac{1000}{\epsilon}}(Q)} + \text{negligible},$$

where the first inequality comes from the fact that $\phi_{(1+2^{\frac{-2\epsilon j}{3}})Q_j,j} = 1$ on the support of $\phi_{Q_j,j}$ and the second inequality comes again from Proposition 5.2.5. Combining all these estimates proves the lemma.

■

Lemma 5.4.3

$$|G_{Q,hh}| \lesssim \sum_{k > j + \frac{1000}{\epsilon}} u_Q 2^{\frac{3j}{2}+k} ||\phi_{Q_j,j} P_k u||_{L^2}^2.$$

Proof We begin similarly to before by estimating

$$\langle \phi_{Q,j} P_j T(P_k u \cdot \nabla \tilde{P}_k u), \phi_{Q,j} P_j u \rangle.$$

With the other terms one can proceed likewise. Now, similarly to what we have already done, we observe that we can write this as,

$$\langle \phi_{Q,j} P_j T(\phi_{Q_j,j} P_k u \cdot \nabla \phi_{Q_j,j} \tilde{P}_k u), \phi_{Q,j} P_j u \rangle + \text{negligible}.$$

Now we estimate

$$||\phi_{Q,j} P_j u||_{L^\infty} \lesssim 2^{\frac{3j}{2}} u_Q + \text{negligible},$$

$$||\nabla \phi_{Q_j,j} \tilde{P}_k u||_{L^2} \lesssim 2^k \sum_{l=-2}^{2} ||\phi_{Q_j,j} P_{k+l} u||_{L^2} + \text{negligible}.$$

Combining these estimates gives the desired inequality. ■

In fact, we get a slightly better estimate, for instance by the div-curl lemma, but it does not seem to be necessary.

The previous three lemmas are somewhat wasteful, particularly in the definition of the local part. We are using more squares than we need to cover. For the result of the chapter 6 we need a somewhat more efficient decomposition, which is proved in exactly the same way. We formulate it here.

For Q a cube at level j, we define the ancestors of Q,

$$\mathcal{A}(Q) = \{Q_k\}_{\epsilon j < k < j-4}.$$

For $k \geq j-4$, we define the collection $\mathcal{S}_k(Q)$ to be a covering (with overlap $\lesssim 1$) of the cube Q_{j-4} by cubes at level k. We define the strict extended family

$$\mathcal{E}(\mathcal{Q}) = \bigcup_{k=j-4}^{j+\frac{1000}{\epsilon}} \mathcal{S}_k(Q),$$

and we define $\mathcal{F}(\mathcal{Q})$, the followers of Q by

$$\mathcal{F}(\mathcal{Q}) = \bigcup_{k>j+\frac{1000}{\epsilon}} \mathcal{S}_k(Q).$$

Then applying the same estimates as in the proofs of lemmas 5.4.1, 5.4.2 and 5.4.3, to a slightly different decomposition we obtain

Corollary 5.4.4 *For any $\delta > 0$ we can estimate*

$$G_Q \lesssim Z_{Q,lhhl} + Z_{Q,loc} + Z_{Q,hh} + 2^{j(1+\frac{3}{2}\delta)} u_Q u_{\mathcal{N}^1(Q)},$$

where

$$Z_{Q,lhhl} = \sum_{k<j-4} \sum_{Q' \in \mathcal{N}^1(Q)} 2^{j+\frac{3k}{2}} u_{Q_k} u_{Q'} u_Q,$$

and

$$Z_{Q,loc} = 2^{\frac{5j}{2}} \sum_{Q',Q'' \in \mathcal{E}(Q)} u_Q u_{Q'} u_{Q''},$$

and finally

$$Z_{Q,hh} = \sum_{k>j+\frac{1000}{\epsilon}} 2^{\frac{3j}{2}+k} u_Q \sum_{Q' \in \mathcal{S}_k(Q)} u_{Q'}^2.$$

Proof This time we define:

$$H'_{j,lh} = \sum_{k<j-4} P_j((P_k u) \cdot \tilde{P}_j \nabla u),$$

$$H'_{j,hl} = \sum_{k<j-4} P_j((\tilde{P}_j u) \cdot P_k \nabla u),$$

$$H'_{j,hh} = \sum_{k>j+\frac{1000}{\epsilon}} P_j((\tilde{P}_k u) \cdot P_k \nabla u) + \sum_{k>j+\frac{1000}{\epsilon}} P_j((P_k u) \cdot \tilde{P}_k \nabla u),,$$

and

$$H'_{loc} = \sum_{j-4<k<j+\frac{1000}{\epsilon}} P_j((\tilde{P}_k u) \cdot P_k \nabla u) + \sum_{j-4<k<j+\frac{1000}{\epsilon}} P_j((P_k u) \cdot \tilde{P}_k \nabla u),.$$

Now we break up G_Q in the obvious way:

$$G_Q = G'_{Q,lh} + G'_{Q,hl} + G'_{Q,hh} + G'_{Q,loc},$$

where

$$G'_{Q,lh} = \langle -\phi_{Q,j} \tilde{P}_j T H'_{j,lh}, \phi_{Q,j} P_j u \rangle,$$

$$G'_{Q,hl} = \langle -\phi_{Q,j} \tilde{P}_j T H'_{j,hl}, \phi_{Q,j} P_j u \rangle,$$

$$G'_{Q,hh} = \langle -\phi_{Q,j} \tilde{P}_j T H'_{j,hh} \phi_{Q,j} P_j u \rangle,$$

$$G'_{Q,loc} = \langle -\phi_{Q,j}\tilde{P}_j T H'_{j,loc}, \phi_{Q,j} P_j u\rangle.$$

Now we simply estimate each of the G''s as before.

■

5.5 Singular set

In this section for each scale j we introduce notion of a bad cube, and a bad set. Intuitively bad cube is a cube on which the nonlinear term dominates the dissipation term, while bad set is a certain union of bad cubes at scale j. On each scale j we shall use Vitali's lemma to cover the corresponding bad set. We shall be able to count elements of these coverings, and that will imply the bound on Hausdorff dimension of our singular set.

Now we are ready to describe our singular set. We will say that a cube Q of sidelength $2^{-j(1-2\epsilon)}$ is **bad** if

$$\int_0^T \int \sum_{k\geq j} 2^{2\alpha k} |\phi_{Q,k} P_k u|^2 \gtrsim 2^{-(5-4\alpha)j-100\epsilon j}. \tag{5.5.1}$$

Let E_j be the union of $2^{\frac{3000}{\epsilon}} Q$ for all cubes Q of sidelength $2^{-j(1-2\epsilon)}$ which are bad.

We will need the following well known covering lemma of Vitali (see (44)):

Lemma 5.5.1 *Let $\mathcal{C}$ be any collection of cubes, then there is a subcollection $\mathcal{C}'$ so that any two cubes in $\mathcal{C}'$ are pairwise disjoint and so that*

$$\bigcup_{Q\in\mathcal{C}} Q \subset \bigcup_{Q\in\mathcal{C}'} 5Q.$$

Proposition 5.5.2 *There is covering $\mathcal{Q}_j$ of E_j by cubes of sidelength $2^{-j(1-2\epsilon)}$ so that*

$$\#(\mathcal{Q}_j) \lesssim 2^{(5-4\alpha)j+100\epsilon j}.$$

Proof Let $\mathcal{C}$ be the collection of cubes $2Q$ where Q is a bad cube at level j. ¿From Lemma 5.5.1 we know that there are disjoint cubes $\{2Q_\rho\}_{\rho\in\mathbb{Z}}$ such that the collection consisting of $\{10Q_\rho\}$ covers the set E_j. Any cube of sidelength $10 \cdot 2^{-j(1-2\epsilon)}$ can be covered by 1000 cubes of sidelength $2^{-j(1-2\epsilon)}$ (and 1000 is a constant.) We will define $\mathcal{Q}_j$ to be the covering formed by the union of the thousandths of the elements of $\{10Q_\rho\}$.

In order to count cubes in $\mathcal{Q}_j$ it is enough to count the disjoint cubes used in the construction of $\mathcal{Q}_j$.

However we know since the $2Q_\rho$'s are disjoint that

$$\sum_\rho \sum_k \int_0^T 2^{2\alpha k} |\phi_{Q_\rho,k} P_k u|^2 \lesssim \int_0^T \int |\Delta^\alpha u|^2 \lesssim 1 \tag{5.5.2}$$

by conservation of energy, while we know

$$\sum_\rho \sum_k \int_0^T 2^{2\alpha k} |\phi_{Q_\rho,k} P_j u|^2 \gtrsim \#(\mathcal{Q}_j) 2^{-(5-4\alpha)j-100\epsilon j} \tag{5.5.3}$$

by the badness of the cubes.

Combining the inequalities (5.5.2) and (5.5.3) implies the claim. ■

We pause for a moment to apply Lemma 4.2.1.

Corollary 5.5.3 *The dimension of* $E = \limsup_{j \longrightarrow \infty} E_j$ *is bounded by* $5 - 4\alpha + O(\epsilon)$.

If we could show that if $x \notin E$ then x is a regular point of u at time T - in other words that

$$\limsup_{t \longrightarrow T} |u(x,t)| < \infty.$$

(Indeed if we could show it is a regular point for a derivative of u of any fixed order) this would immediately imply Theorem 1.2.2.

CHAPTER 6

CRITICAL REGULARITY FOR THE NAVIER-STOKES EQUATIONS WITH HYPER-DISSIPATION

6.1 Introduction

We continue to consider the Navier-Stokes equations with hyper-dissipation. In this chapter for each scale of frequency localization j, we prove a certain level of regularity, which is valid away from the corresponding bad set. This level of regularity shall express a balance between the nonlinear and the dissipation terms as in the case of the dyadic Navier-Stokes equations with hyper-dissipation.

Let E be defined as in the previous chapter, i.e.

$$E = \limsup_{j \longrightarrow \infty} E_j,$$

where E_j stands for the bad set at scale j.

First we investigate what are the immediate consequences of $x \notin E$. Saying that $x \notin E$ is the same as saying that there exists a j so that so that for any $k > j$, we have $x \notin E_k$. Denote F_j as the set of points with this property. To prove a regularity statement about E, it suffices to show that that statement holds for any j_0, provided $x \in F_{j_0}$. However fixing j, we may change our constants in the definition of bad square so that x is not contained in any bad squares. Thus we may as well assume x is contained in no bad squares. This will be our hypothesis in this chapter. (And, the constants will now depend on j_0.)

Now we would like to introduce a level of regularity which intuitively separates the bad set from its complement. As in the dyadic heuristic as long as the dissipation term dominates the nonlinear term the growth of u_Q should be under control. Heuristically, the worst part of the nonlinear part G_Q is $G_{Q,loc}$ which looks at scale j like $2^{\frac{5j}{2}} u_Q^3$. On the other hand, dissipation gives decay like $2^{2\alpha j} u_Q^2$. This should mean that as long as $u_Q < 2^{2(\alpha - \frac{5}{4})j}$, the growth of u_Q ought to be under control. It is this estimate which we will show in the following section for any Q not contained in E_k for any $k > j_0$.

6.2 Critical regularity theorem

Theorem 6.2.1 *Let Q be as above, then there is a constant C, depending only on T, the initial conditions for u, the constant in the definition of badness, and j_0 so that*

$$u_Q(t) < C 2^{2(\alpha - \frac{5}{4} - \frac{\epsilon}{2}) j(Q)}.$$

Proof We proceed by contradiction. Suppose the theorem is false. We let T_0 be the first time and Q be the largest cube so that

$$u_Q(T_0) > C 2^{2(\alpha - \frac{5}{4} - \frac{\epsilon}{2}) j}.$$

Now since our initial data is smooth, at the initial time, (since j is chosen sufficiently large), we have

$$u_Q(0) \lesssim 2^{-1000j}. \tag{6.2.1}$$

Recall that in chapter 5 we wrote a localized version of the energy equation for the Navier-Stokes equation in the form:

$$\frac{1}{2}\frac{d}{dt}u_Q^2 = \langle -T(u\cdot\nabla u), P_j\phi_{Q,j}^2 P_j u\rangle - \langle(-\Delta)^\alpha u, P_j\phi_{Q,j}^2 P_j u\rangle,$$

which was by introducing G_Q rewritten as

$$\frac{1}{2}\frac{d}{dt}u_Q^2 = G_Q - \langle(-\Delta)^\alpha u, P_j\phi_{Q,j}^2 P_j u\rangle.$$

Therefore having in mind the lower bound on the dissipation term

$$\langle(-\Delta)^\alpha u, P_j\phi_{Q,j}^2 P_j u\rangle \geq c2^{2\alpha j}u_Q^2 - C2^{(2\alpha-2\epsilon)j}\sum_{Q'\in\mathcal{N}^1(Q)} u_{Q'}^2 - \text{negligible}$$

as well as (6.2.1), it must be the case that

$$\int_0^{T_0}(G_Q(t) - c2^{2\alpha j}u_Q^2 + C2^{(2\alpha-2\epsilon)j}\sum_{Q'\in\mathcal{N}^1(Q)} u_{Q'}^2)dt \gtrsim 2^{4(\alpha-\frac{5}{4}-\frac{\epsilon}{2})j}.$$

By using the "good neighbors" proposition, we can replace Q by an extended nuclear family $\mathcal{N}^l(Q)$, with $l < \frac{400}{\epsilon}$ for which

$$\int_0^T(-c2^{2\alpha j}u_{\mathcal{N}^l(Q)}^2 + C2^{(2\alpha-2\epsilon)j}u_{\mathcal{N}^{l+1}(Q)}^2)dt \lesssim -\int_0^T 2^{2\alpha j}u_{\mathcal{N}^l(Q)}^2.$$

For the current theorem, this is all we need. Since $u_{\mathcal{N}^l(Q)}$ also begins $\lesssim 2^{-1000j}$, we must have

$$\int_0^{T_0} G_{\mathcal{N}^l(Q)}(t)dt \gtrsim 2^{4(\alpha-\frac{5}{4}-\frac{\epsilon}{2})j} + \int_0^T 2^{2\alpha j} u^2_{\mathcal{N}^l(Q)},$$

where we define

$$G_{\mathcal{N}^l(Q)}(t) = \sum_{Q_1 \in \mathcal{N}^l(Q)} G_{Q_1}(t).$$

Since there are only $\lesssim 1$ cubes in $\mathcal{N}^l(Q)$, for this to be the case, there must be a $\tilde{Q}$ so that

$$\int_0^{T_0} G_{\tilde{Q}}(t)dt \gtrsim 2^{4(\alpha-\frac{5}{4}-\frac{\epsilon}{2})j} + \int_0^T 2^{2\alpha j} u^2_{\mathcal{N}^l(\tilde{Q})},$$

We contradict this by using Lemmas 5.4.1, 5.4.2, and 5.4.3 to estimate $G_{\tilde{Q}}$. By our definition of E_j, the cube $\tilde{Q}$ is contained in no larger bad squares.

Each estimate contains a factor of $u_{\tilde{Q}}$ which we take out using the estimate $|u_{\tilde{Q}}| \lesssim 2^{2(\alpha-\frac{5}{4}-\frac{\epsilon}{2})j}$, which we get from the definition of T_0.

First let Q_1 be a nuclear family member of $\tilde{Q}$ and Q_2 be a distant ancestor at level k. Suppose $k < \delta j$, we see that $2^{(1+\frac{3\delta}{2})j} u_{Q_1} \lesssim 2^{2\alpha j} u_{\mathcal{N}^l(\tilde{Q})}$. Thus we need not worry about this term.

Suppose $k > \delta j$. We must estimate

$$\int_0^T 2^{\frac{3k}{2}+j} u_{Q_1} u_{Q_2} dt.$$

Note that both Q_1 and Q_2 are good squares. Thus we have the estimates

$$\int_0^T 2^{3k} u^2_{Q_2} \leq 2^{(2\alpha-2)k-10\epsilon k},$$

and

$$\int_0^T 2^{2j} u_{Q_1}^2 \leq 2^{(2\alpha-3)j-10\epsilon j}.$$

Applying Cauchy-Schwartz, we get (using $\alpha \geq 1$)

$$\int_0^T 2^{\frac{3k}{2}+j} u_{Q_1} u_{Q_2} dt \leq 2^{(\frac{4\alpha-5}{2})j-10\epsilon j}.$$

Summing over (there are only j terms) provides the desired estimate on $G_{\tilde{Q},hl}+G_{\tilde{Q},lh}$. We can estimate $G_{\tilde{Q},loc}$ in the same way (by allowing k as large as $j+\frac{1000}{\epsilon}$.)

We are left to estimate $G_{\tilde{Q},hh}$. We fix a scale $k > j$ and pick k_2 within 2 of k. Now we are left to estimate

$$\int_0^{T_0} 2^{\frac{3j}{2}+k} ||\phi_{\tilde{Q},k} P_k u||_{L^2} ||\phi_{\tilde{Q},k_2} P_{k_2} u||_{L^2}.$$

However since $\tilde{Q}$ is a good square, we have

$$\int 2^{2\alpha k} |\phi_{\tilde{Q},k} P_k u|^2 \leq 2^{(4\alpha-5)j-10\epsilon j}.$$

Thus

$$\int 2^{k+\frac{3j}{2}} |\phi_{\tilde{Q},k} P_k u|^2 \leq 2^{(\frac{4\alpha-5}{2})j-10\epsilon j-(2\alpha-1)(k-j)},$$

which is an estimate that decays geometrically in k when $\alpha > \frac{1}{2}$. By using the similar estimate for k_2, applying Cauchy Schwartz and summing over k, we get the desired result. ■

CHAPTER 7

BARRIER ESTIMATE FOR THE NAVIER-STOKES EQUATIONS WITH HYPER-DISSIPATION

7.1 Introduction

In this chapter we finish the proof of the main theorem by proving that a solution of the Navier-Stokes equations with hyper-dissipation is regular outside of a certain bad set. Ideally we would like to prove that if $x \notin E$ then

$$\limsup_{t \longrightarrow T} |u(x,t)| < \infty.$$

However due to a combinatorial issue we shall prove the same statement for a somewhat larger collection than one which covers the set E.

We shall prove regularity far inside a cube Q provided that one has a little better than critical regularity for the cube Q as well as smooth initial data. By "a little better than critical regularity for the cube Q" we mean:

- critical regularity for all cubes containing our cube Q, and
- critical regularity for all boundary cubes of the cube Q.

The last one which we could think of as a "safe boundary condition" imposes the restriction on α which is $\alpha > 1$.

Now we shall prove a statement which verifies the "safe boundary condition". We do that in section 7.2. Then in section 7.3 we prove a barrier estimate which guarantees regularity far inside the cube Q.

7.2 Combinatorics

Here we shall verify that we can replace our cube Q with a slightly smaller cube whose boundary is away from the singular set. Let us think heuristically about the boundary of our cube Q. Since we are in $\mathbb{R}^3$ the boundary of the cube Q is 2-dimensional. However the dimension of the singular set is $5 - 4\alpha$ and this number is smaller than 1, provided that $\alpha > 1$. Now we need to assure that 2-dimensional boundary does not intersect the singular set whose dimension is less than 1. We can do that because we are in 3 dimensions. In other words we will replace our cube Q with slightly smaller cube with safe boundary. More precisely, we prove the theorem which guarantees the existence of a number r, $\frac{1}{2} < r < 1$ such that the cube rQ would have a nice boundary, i.e. boundary which is away from a certain bad set.

We begin with the set E_j which is the union of a collection $\mathcal{Q}_j$ of cubes with sidelength $2^{-(1-2\epsilon)j}$ having cardinality $\lesssim 2^{(5-4\alpha+100\epsilon)j}$. (We assume $4\alpha - 4 > 200\epsilon$.)

Theorem 7.2.1 *There exists a sequence of collections $\mathcal{Q}'_j$ of cubes of sidelength $2^{-(1-2\epsilon)j}$ with $\#(\mathcal{Q}'_j) \lesssim 2^{(5-4\alpha+100\epsilon)j}$, so that for any Q of length $2^{-(1-2\epsilon)j}$ which does not intersect any element $\mathcal{Q}'_j$ there exists a number $\frac{1}{2} < r < 1$ with the following property: For no $k > j$ is there $\tilde{Q} \in \mathcal{Q}_k$ so that*

$$100\tilde{Q} \cap \partial(rQ) \neq \emptyset \tag{7.2.1}$$

Proof We refer to the elements of $\mathcal{Q}_j$ as the bad cubes. We say that a cube of sidelength $2^{-(1-2\epsilon)j}$ is very bad if either it intersects a bad cube of the same length or it intersects more than $c2^{(5-4\alpha+150\epsilon)(k-j)}$ elements of $100\mathcal{Q}_k$ for some $k > j$ with c a small constant to be specified later. Let E'_j be the union of all very bad cubes of length $2^{-(1-2\epsilon)j}$, then by the estimates on the cardinality of the $\mathcal{Q}_k$'s and by the Vitali lemma, we can see that that E_j' can be covered by $\lesssim 2^{(5-4\alpha+100\epsilon)j}$ cubes of length $2^{-(1-2\epsilon)j}$. We refer to these cubes as $\mathcal{Q}_k'$. Now we need only prove (7.2.1).

Let Q be a cube of length $2^{-(1-2\epsilon)j}$ which does not intersect E_j'. Let $\mathcal{D}^k(Q)$ be the set of elements of $100\mathcal{Q}_k$ which intersect Q. Then we have the estimate

$$\#(\mathcal{D}^k(Q)) \leq c2^{(5-4\alpha+150\epsilon)(k-j)}.$$

Let $f_k(r)$ be the function defined from $\frac{1}{2}$ to 1 which counts how many elements of $\mathcal{D}^k(Q)$ intersect $\partial(rQ)$. For each $Q' \in \mathcal{D}^k(Q)$ define $r_{Q'}$ to be that number so that the center of Q' lies on $\partial(r_{Q'}Q)$. Then

$$f_k(r) \leq \sum_{Q' \in \mathcal{D}^k(Q)} \chi_{(r_{Q'}-100(2^{(1-2\epsilon)(j-k)}), r_{Q'}+100(2^{(1-2\epsilon)(j-k)}))}.$$

Thus

$$||f_k(r)||_{L^1} \leq 200c2^{(4\alpha-4-152\epsilon)(j-k)}.$$

Since $4\alpha - 4 > 200\epsilon$, this estimate decays geometrically with k. By choosing c sufficiently small, we may arrange that

$$||\sum_{k>j} f_k||_{L^1} < \frac{1}{4}.$$

Thus by Tchebychev's inequality,

$$|\{r : |\sum_{k>j} f_k(r)| \geq 1\}| \leq \frac{1}{4}.$$

Therefore having in mind that $f_k(r)$ is integer valued, we conclude that there must be a value of r between $\frac{1}{2}$ and 1 so that $f_k(r) = 0$ for all $k > j$. This is the value of r that we choose. ■

7.3 Barrier estimate

In this section, we prove regularity on the interior of a cube Q, provided that one has critical regularity for cubes containing it and cubes Q' for which $\partial Q \cap 100Q' \neq \emptyset$.

If Q is a cube and $Q_1 \subset Q$, we define $d(Q_1)$, the graph distance of Q_1 to the boundary of Q by $d(Q_1) = k - 1$, where k is the smallest positive integer so that

$$2^k Q_1 \cap \partial Q \neq \emptyset.$$

Lemma 7.3.1 *Let Q be a cube. Suppose we know that for all $t < T$ we have that for any cube Q' so that $Q \subset Q'$ with sidelength of Q' being $2^{-l(1-2\epsilon)}$, we have that*

$$|u_{Q'}(t)| \lesssim 2^{-l(\frac{5-4\alpha+2\epsilon}{2})},$$

and suppose further that for any Q' so that $100Q' \cap \partial Q \neq \emptyset$ with sidelength of Q' being $2^{-l(1-2\epsilon)}$ and $l > j - 2$, we have that

$$|u_{Q'}(t)| \lesssim 2^{-l(\frac{5-4\alpha+2\epsilon}{2})},$$

then for any $Q_1 \subset Q$ of length $2^{-k(1-2\epsilon)}$, we have the estimate

$$|u_{Q_1}(t)| \lesssim 2^{-k\rho(Q_1)}, \tag{7.3.1}$$

where

$$\rho(Q_1) = \min(10, \frac{5-4\alpha+2\epsilon}{2} + \frac{\epsilon(d(Q_1)-5)}{50}).$$

Proof The proof we present is by contradiction.

Notice that for all cubes Q_1 of sidelength $2^{-(1-2\epsilon)k}$ we have

$$|u_{Q_1}(0)| \lesssim 2^{-1000k}, \tag{7.3.2}$$

i.e. the lemma is satisfied at time $t = 0$.

Let t_1 be the first time at which the lemma fails and Q_1 be one of the cubes for which it fails. It must be the case by hypothesis that $32Q_1$ does not intersect ∂Q. Then we have

$$u_{Q_1}(t_1) \sim 2^{-k\rho(Q_1)}.$$

Having in mind (7.3.2), we can find the time t_0, being the last time before t_1 when

$$u_{Q_1}(t_0) \lesssim 2^{k(-\rho(Q_1)+\frac{\epsilon}{10})}.$$

Then we have

$$\int_{t_0}^{t_1} \frac{d}{dt}(u_{Q_1}^2) \gtrsim 2^{-2k\rho(Q_1)}. \tag{7.3.3}$$

However on the time interval (t_0, t_1) the lemma is satisfied.

We will invoke Corollary 5.4.4. Now for any $Q_2 \in \mathcal{E}(Q_1)$ we have

$$d(Q_2) \geq d(Q_1) - 5. \tag{7.3.4}$$

This is because $Q_2 \subset 32Q_1$. Further for any ancestor $Q_3 \in \mathcal{A}(Q_1)$ with Q_3 having sidelength $2^{-(1-2\epsilon)l}$, with $\epsilon k < l < k-4$,

$$d(Q_3) \geq d(Q_1) - 5(k-l). \tag{7.3.5}$$

Further for any follower $Q_4 \in \mathcal{F}(Q_1)$ (that is a cube which contributes to $G_{Q_1,hh}$ and is in particular contained in $\frac{3}{2}Q_1$) of Q_1 with Q_4 having sidelength $2^{-(1-2\epsilon)l}$ with $l > k + \frac{1000}{\epsilon}$, we have

$$d(Q_4) \geq d(Q_1) + (\frac{l-k}{2}). \tag{7.3.6}$$

Applying (7.3.6), we conclude that $\rho(Q_4) = 10$. Thus we conclude

$$Z_{Q_1,hh} \lesssim \sum_{l>k} 2^{\frac{5k}{2}} 2^{3(l-k)} 2^{-20l} 2^{(-\rho(Q_1)+\frac{\epsilon}{10})k},$$

by counting the elements of $\mathcal{S}_l(Q)$ by $2^{3(l-k)}$. Calculating, we find

$$Z_{Q_1,hh} \lesssim 2^{-(\frac{35}{2}+\rho(Q_1))k},$$

which since $\rho(Q_1) \leq 10$, cannot possibly account for (7.3.3).

Now applying (7.3.4) to Proposition 5.3.1, we observe that for any t between t_0 and t_1, we have dissipation at Q_1 of

$$\gtrsim 2^{2\alpha k} u_{Q_1}^2(t) \gtrsim 2^{(2\alpha - 2\rho(Q_1) + \frac{\epsilon}{5})k}.$$

Thus to reach a contradiction, it suffices to show that on this time interval

$$Z_{Q_1,hllh} + Z_{Q_1,loc} \lesssim 2^{(2\alpha - 2\rho(Q_1) + \frac{\epsilon}{5})k}.$$

For this reason we can also ignore the ultra-low term $2^{j(1+\frac{3}{2}\delta)} u_{Q_1} u_{\mathcal{N}^1(Q_1)}$ in Corollary 5.4.4.

In the rest of the proof we shall use the advantage of the fact that the lemma is satisfied on the time interval (t_0, t_1). First notice that by (7.3.1) we have $|u_{Q_1}| < 2^{-k\rho(Q_1)}$, for all $t \in [t_0, t_1]$.

Using (7.3.1) and (7.3.4), we observe that for any Q_2 in $\mathcal{E}(Q_1)$ and for all $t \in [t_0, t_1]$ we have

$$\begin{aligned} u_{Q_2} &\lesssim 2^{-k\rho(Q_2)} \\ &= 2^{-k(\frac{5-4\alpha+2\epsilon}{2} + \frac{\epsilon(d(Q_2)-5)}{50})} \\ &\leq 2^{-k(\frac{5-4\alpha+2\epsilon}{2} + \frac{\epsilon(d(Q_1)-5)}{50}) + \frac{k\epsilon}{10}} \\ &= 2^{-k\rho(Q_1) + \frac{k\epsilon}{10}}, \end{aligned}$$

and therefore

$$u_{Q_2} \lesssim 2^{(-\rho(Q_1) + \frac{\epsilon}{10})k}. \tag{7.3.7}$$

On the other hand for any Q_2 in $\mathcal{E}(Q_1)$ we have

$$u_{Q_2} \lesssim 2^{\frac{(4\alpha-5-2\epsilon)k}{2}}, \tag{7.3.8}$$

by the lower bound on ρ.

Thus by using (7.3.7) and (7.3.8) we bound $Z_{Q_1,loc}$

$$Z_{Q_1,loc} \lesssim \sum_{Q_2,Q_2'\in\mathcal{E}(Q_1)} 2^{\frac{5k}{2}} u_{Q_2} u_{Q_2'} u_{Q_1} \lesssim 2^{(2\alpha-\frac{4\epsilon}{5}-2\rho(Q_1))k}.$$

Thus $Z_{Q_1,loc}$ cannot contribute to the growth.

Now to estimate $Z_{Q_1,hllh}$, we observe that for $Q_1' \in \mathcal{N}^1(Q_1)$ we have the estimate

$$u_{Q_1'} \lesssim 2^{(-\rho(Q_1)+\frac{\epsilon}{10})k},$$

while for ancestor Q_3 of sidelength $2^{-(1-2\epsilon)l}$, we apply (7.3.5) (as well as the hypotheses of the lemma for squares larger than Q)

$$u_{Q_3} \lesssim 2^{\frac{(4\alpha-5-2\epsilon)k}{2}+(\frac{3}{2}-100\epsilon)(k-l)},$$

since of course $\frac{3}{2} > \frac{5-4\alpha}{2} + 200\epsilon$. Now we just estimate

$$Z_{Q_1,hllh} \lesssim \sum_l \sum 2^{\frac{3l}{2}+k} u_{Q_1'} u_{Q_3} u_{Q_1} \lesssim 2^{(2\alpha-\frac{4\epsilon}{5}-2\rho(Q_1))k}.$$

Thus u_{Q_1} could not have grown which is a contradiction. ∎

But now we have in fact proven the main theorem. We define E'_j as in section 7.2. We need to show that u is regular at any x not contained in any E'_j with j larger than some integer j_x. By changing our constants, we can say x is not in any E'_j. Now by theorems 6.2.1 and 7.2.1, for any cube Q centered at x, we can find a cube almost half as large which satisfies the hypotheses of Lemma 7.3.1. But the conclusion of the lemma implies that x is a regular point. Thus any singular point must be contained in $E = \limsup E'_j$. By lemma 4.2.1, we have $\dim(E) < 5 - 4\alpha + 20\epsilon$. Letting ϵ tend towards 0, we get Theorem 1.2.2.

CHAPTER 8

DYADIC MODELS AND BLOW-UP RESULTS

8.1 Introduction

In this chapter we revisit the dyadic model for the Euler equations and the Navier-Stokes equations with hyper-dissipation in three dimenison. For the dyadic Euler equations we prove finite time blow-up. In the context of the dyadic Navier-Stokes equations with hyper-dissipation we prove finite time blow-up in case when the dissipation degree is sufficiently small.

For both the dyadic Euler and the dyadic Navier-Stokes equations with hyper-dissipation we prove an estimate which gives a lower bound on the nonlinear term. Notice that in the context of partial regularity results presented in previous chapters we always used an upper bound on the nonlinear term. Then we calculated a balance betwwen the nonlinear and the dissipation term. However for the blow-up results one needs a lower bound on the nonlinear term. Such a bound describes concentration of energy sufficient to produce a blow-up. We prove for the dyadic model that a lower bound guarantees blow-up, provided that the dissipation degree is small. However we are not able to produce such a lower bound for the actual equations themselves.

Before going into details let us recall the model. Following the notation introduced in chapter 4, we use the bilinear operator $C(u,v)$, which is built from two pieces, $C_u(u,v)$ and $C_d(u,v)$, and we have:

$$(C_d(u,u))_Q = 2^{\frac{5j(Q)}{2}} u_{\tilde{Q}}^2,$$

$$(C_u(u,u))_Q = 2^{\frac{5(j(Q)+1)}{2}} u_Q \sum_{Q' \in C^1(Q)} u_{Q'},$$

$$C(u,u) = C_u(u,u) - C_d(u,u).$$

Clearly we always have antisymmetry in the sense that

$$\langle C(u,u), u \rangle = 0.$$

We shall say that a time varying "function" u satisfies the dyadic Euler equation provided that

$$\frac{du}{dt} + C(u,u) = 0. \tag{8.1.1}$$

Also we shall say that u satisfies the dyadic Navier-Stokes equations with hyper-dissipation if

$$\frac{du}{dt} + C(u,u) + (\Delta)^{\alpha} u = 0. \tag{8.1.2}$$

We will restrict our attention to "functions" u all of whose coefficients u_Q are initially positive. This class of functions is preserved by both flows (8.1.1) and (8.1.2). Let us verify that for the flow (8.1.1). Fix a dyadic cube Q. We rewrite the equation (8.1.1) in terms of wavelet coefficients as follows:

$$\frac{du_Q(t)}{dt} + 2^{\frac{5j(Q)}{2}} u_Q \sum_{Q' \in C^1(Q)} u_{Q'}(t) = u_{\tilde{Q}}^2(t). \tag{8.1.3}$$

Now we remark that the equation (8.1.3) is a first order linear ordinary differential equation in $u_Q(t)$ and its solution is:

$$u_Q(t) = \frac{1}{\mu(t)}(u_Q(0) + \int_0^t u_{\tilde{Q}}^2(\tau)\mu(\tau)d\tau), \tag{8.1.4}$$

where

$$\mu(t) = e^{\int_0^t 2^{\frac{5j(Q)}{2}} \sum_{Q' \in C^1(Q)} u_{Q'}(\tau)d\tau}.$$

Since $\mu(t) > 0$ for all t, (8.1.4) implies that $u_Q(t) > 0$ for all $t > 0$, provided that $u_Q(0)$ is positive.

The chapter is organized as follows. In section 8.2 we give a blow-up result for the dyadic Euler equation. Then in section 8.3 we present a proof of finite time blow-up for the dyadic Navier-Stokes equations with small dissipation.

8.2 The dyadic Euler equation

8.2.1 Energy flow

One of the most important features of the flow (8.1.1) is that it conserves energy. To be more precise

$$\frac{d}{dt}(\langle u, u\rangle) = 0 \tag{8.2.1}$$

This can be obtained by pairing (8.1.1) with u. Energy can be thought of as divided up amongst the nodes Q. To be more precise, if we write

$$E = \langle u, u\rangle,$$

then

$$E = \sum_{Q \in \mathcal{D}} E_Q,$$

where

$$E_Q = u_Q^2.$$

The flow (8.1.1) gives rise to an extremely local description of energy flow along the tree $\mathcal{D}$.

To be precise

$$\frac{d}{dt} E_Q = E_{Q,in} - E_{Q,out}, \tag{8.2.2}$$

where

$$E_{Q,in} = 2^{\frac{5j(Q)}{2}} (E_{\tilde{Q}} \sqrt{E_Q}),$$

and

$$E_{Q,out} = \sum_{Q' \in \mathcal{C}^1(Q)} E_{Q',in}.$$

Thus energy is flowing always from larger squares to smaller ones and indeed it flows along the edges of the tree $\mathcal{D}$.

We define a Carleson box by

$$\mathcal{C}(Q) = \bigcup_{k=1}^{\infty} \mathcal{C}^k(Q),$$

and the energy of a Carleson box by

$$E_{\mathcal{C}(Q)} = \sum_{Q' \in \mathcal{C}(Q)} E_{Q'}.$$

Also we shall introduce an extended Carleson box by

$$\mathcal{C}_0(Q) \equiv Q \cup \mathcal{C}(Q).$$

Then we define the energy of $\mathcal{C}_0(Q)$ by

$$E_{\mathcal{C}_0(Q)} = \sum_{Q' \in \mathcal{C}_0(Q)} E_{Q'},$$

We immediately get the following proposition.

Proposition 8.2.1.1 *Let u be a time-varying "function" with positive coefficients evolving according to the flow* (8.1.1)*. Then for any Q, the functions in time given by $E_{\mathcal{C}(Q)}$ and $E_{\mathcal{C}_0(Q)}$ are monotone increasing.*

8.2.2 The heart

We begin with an easy lemma about Carleson boxes.

Lemma 8.2.2.1 *For any $\epsilon > 0$, there is $\delta(\epsilon) > 0$, so that if we know that*

$$E_{\mathcal{C}(Q)} > (1 - \delta) 2^{-(3+\epsilon) j(Q)},$$

then there exists $Q' \in \mathcal{C}(Q)$, *so that*

$$E_{\mathcal{C}_0(Q')} \geq 2^{-(3+\epsilon)j(Q')}.$$

Proof Let us suppose the conclusion of the lemma is false, i.e.

$$E_{\mathcal{C}_0(Q')} < 2^{-(3+\epsilon)j(Q')}, \tag{8.2.3}$$

for all $Q' \in \mathcal{C}(Q)$.

On the other hand we know

$$E_{\mathcal{C}(Q)} = \sum_{Q' \in \mathcal{C}^1(Q)} E_{\mathcal{C}_0(Q')}. \tag{8.2.4}$$

Now because we are in dimension 3, there are exactly 2^3 elements $Q' \in \mathcal{C}^1(Q)$ with $j(Q') = j(Q)+1$. Thus(8.2.4) combined with (8.2.3) implies

$$E_{\mathcal{C}(Q)} \leq 2^3 \cdot 2^{-(3+\epsilon)(j(Q)+1)},$$

and therefore

$$E_{\mathcal{C}(Q)} \leq 2^{-\epsilon} \cdot 2^{-(3+\epsilon)j(Q)}$$

which is a contradiction provided we have chosen δ sufficiently small that $2^{-\epsilon} < 1 - \delta$. ∎

Now we prove the main lemma.

Lemma 8.2.2.2 *Fix j_0 sufficiently large. Then there is a sufficiently small $0 < \epsilon < 1$ so that if at time t_0, we have*

$$E_{\mathcal{C}_0(Q)} \geq 2^{-(3+\epsilon)j(Q)}, \tag{8.2.5}$$

with $j(Q) > j_0$, then there is some t with $t < t_0 + 2^{-\epsilon j(Q)}$ and a cube $Q' \in \mathcal{C}(Q)$ so that at time t, we have $E_{\mathcal{C}_0(Q')} \geq 2^{-(3+\epsilon)j(Q')}$.

Proof We assume that the conclusion of the lemma is false, i.e.

$$E_{\mathcal{C}_0(Q')} < 2^{-(3+\epsilon)j(Q')},$$

for all $Q' \in \mathcal{C}(Q)$ and for all $t \in [t_0, t_0 + 2^{-\epsilon j(Q)}]$.

In light of Lemma 8.2.2.1 and Proposition 8.2.1.1, it must be the case that for all $t \in [t_0, t_0 + 2^{-\epsilon j(Q)}]$ we have $E_Q \geq \delta 2^{-(3+\epsilon)j(Q)}$, since otherwise because of the hypothesis (8.2.5) we would have $E_{\mathcal{C}(Q)} > (1-\delta)2^{-(3+\epsilon)j(Q)}$, which would by Lemma 8.2.2.1 lead to a contradiction.

Moreover since energy flows only in the direction of smaller squares and is conserved, it must be that for any child $Q' \in \mathcal{C}^1(Q)$, it is the case that

$$E_{\mathcal{C}(Q)} \geq \int_{t_0}^{t_0 + 2^{-\epsilon j(Q)}} E_{Q',in},$$

at time $t_0 + 2^{-\epsilon j(Q)}$. Thus since

$$E_{Q',in} = 2^{\frac{5(j(Q)+1)}{2}} E_Q u_{Q'},$$

we must have

$$\int_{t_0}^{t_0+2^{-\epsilon j(Q)}} u_{Q'} \lesssim \frac{1-\delta}{\delta} 2^{\frac{-5j(Q)}{2}}. \tag{8.2.6}$$

However we know that

$$\frac{du_{Q'}}{dt} = 2^{\frac{5(j(Q)+1)}{2}} E_Q - 2^{\frac{5(j(Q)+2)}{2}} u_{Q'} \Big(\sum_{Q'' \in \mathcal{C}^1(Q')} u_{Q''} \Big). \tag{8.2.7}$$

Thus integrating, applying the fact that for all $t \in [t_0, t_0 + 2^{-\epsilon j(Q)}]$, we have $E_Q > \delta 2^{-(3+\epsilon) j(Q)}$, and $u_{Q''} \lesssim 2^{-\frac{3j(Q)}{2}}$ (we can afford to give back the ϵ), and using (8.2.6), we see that the first term of (8.2.7) dominates (for ϵ sufficiently large and $j(Q)$ sufficiently small) and

$$\int_{t_0}^{t_0+2^{-\epsilon j(Q)}} \frac{du_{Q'}}{dt} \gtrsim \delta 2^{-(\frac{1}{2}+2\epsilon) j(Q)},$$

which is a contradiction by the fundamental theorem of calculus.

■

Corollary 8.2.2.3 *Let u be a solution to* (8.1.1) *which has initially all positive coefficients and at time 0, has $E_Q > 2^{-(3+\epsilon) j(Q)}$, for $j(Q) > j_0$ with j_0 as in the previous lemma. Then the $H^{\frac{3}{2}+\epsilon}$ norm of u becomes unbounded in finite time.*

Proof We apply the lemma. We find a cube Q_1 properly contained in Q and a time $t_1 < 2^{-\epsilon j(Q)}$ so that at t_1 we have $E_{\mathcal{C}_0(Q_1)} > 2^{-(3+\epsilon) j(Q_1)}$.

We iterate this procedure finding a cube Q_k properly contained in Q_{k-1} and and a time t_k so that $t_{k-1} \leq t_k < t_{k-1} + 2^{-\epsilon j(Q_{k-1})}$ and at time t_k we have $E_{\mathcal{C}_0(Q_k)} > 2^{-(3+\epsilon)j(Q_k)}$.

Estimating just using the coefficient at $\mathcal{C}_0(Q_k)$, we see that at time t_k, we have that

$$||u||_{H^{\frac{3}{2}+\epsilon}} \geq 2^{\frac{\epsilon j(Q_k)}{2}}.$$

Since $j(Q_k)$ is an increasing sequence of integers, this is going to ∞. However since,

$$t_k = (t_k - t_{k-1}) + (t_{k-1} - t_{k-2}) + \cdots + t_1 \leq 2^{-\epsilon j(Q)} + \sum_{l=1}^{k-1} 2^{-\epsilon j(Q_l)},$$

and the $j(Q_l)$'s are an increasing sequence of integers, we see that the sequence $\{t_k\}$ converges to a finite limit. ■

We point our here that under certain assumptions on the initial $E_Q(0)$ Corollary 8.2.2.3 guarantees finite time blow-up of $||u||_{H^{\frac{3}{2}+\epsilon}}$-norm. This was obtained using an extended Carleson box $\mathcal{C}_0(Q)$. In the next section we shall prove for the Navier-Stokes equations with small dissipation, finite time blow-up of slightly weaker $||u||_{H^{2+\epsilon}}$-norm. The reason for this is the fact that for the Navier-Stokes equations with hyper-dissipation the energy itself is not conserved. Instead part of it dissipates at each level of our dyadic tree $\mathcal{D}$.

However from Fedor Nazarov (Nazarov, 2001, personal communication) we learned his proof that the solution to the dyadic Navier-Stokes equations with enough dissipation stays bounded in a certain C^k space provided that it started in the same C^k space. The main tool in his proof is the following observation. Let us truncate the system of ODEs which describe the dyadic Navier-Stokes equations with hyper-dissipation. If provided with enough dissipation the system will scatter all energy. Therefore energy cannot become concentrated over first few levels. For example, this is true for the dyadic Navier-Stokes itself. The natural question is what does "enough dissipation" mean. In the following section we prove that in order to have finite time blow-up the dissipation exponent α should be less than $\frac{1}{4}$.

8.3 The dyadic Navier-Stokes equations with hyper-dissipation

8.3.1 Energy flow

We consider the dyadic Navier-Stokes equations with hyper-dissipation (8.1.2). Since

$$\langle C(u,u),u\rangle = 0,$$

we have:

$$\frac{d}{dt}(\langle u,u\rangle) + \langle (\Delta)^{\alpha}u,u\rangle = 0,$$

and therefore we have energy decay:

$$\langle u,u\rangle + \int_0^t \langle (\Delta)^{\alpha}u,u\rangle = 0. \tag{8.3.1}$$

Let us imagine that each node Q along our tree $\mathcal{D}$ has a wastebasket which is at time t filled with $\int_{t_0}^t 2^{2\alpha j(Q)}u_Q^2$.

We define the energy of a cube Q at time t as in the case of the dyadic Euler equation

$$E_Q(t) = u_Q^2(t).$$

For t greater than or equal to some fixed time t_0 we introduce the energy of a wastebasket of a cube Q at time t as

$$W_{Q,t_0}(t) = \int_{t_0}^{t} 2^{2\alpha j(Q)} E_Q.$$

Then (8.3.1) is saying that the sum of energy at nodes plus energy at wastebaskets is controlled.

Also we have a local description of energy flow along the tree $\mathcal{D}$:

$$\frac{d}{dt}E_Q = E_{Q,in} - E_{Q,out} - 2^{2\alpha j(Q)} E_Q, \tag{8.3.2}$$

where

$$E_{Q,in} = 2^{\frac{5j(Q)}{2}} E_{\tilde{Q}} \sqrt{E_Q},$$

and

$$E_{Q,out} = \sum_{Q' \in \mathcal{C}^1(Q)} E_{Q',in}.$$

Thus energy is flowing always from larger squares to smaller ones and, except for portions which go to wastebaskets, energy flows along the edges of the tree $\mathcal{D}$.

We define the energy of a Carleson box by

$$E_{\mathcal{C}(Q)} = \sum_{Q' \in \mathcal{C}(Q)} E_{Q'},$$

and the waste of a Carleson box by

$$W_{\mathcal{C}(Q),t_0} = \sum_{Q' \in \mathcal{C}(Q)} W_{Q',t_0}.$$

We immediately get the following proposition.

Proposition 8.3.1.1 *Let u be a time-varying "function" with positive coefficients evolving according to the flow* (8.1.2)*. Then for any Q, the function in time given by $E_{\mathcal{C}(Q)} + W_{\mathcal{C}(Q),t_0}$ is monotone increasing.*

8.3.2 Energy concentration

As in the Euler case we begin with a lemma about Carleson boxes.

Lemma 8.3.2.1 *For any $\epsilon > 0$, there is $\delta(\epsilon) > 0$, so that if we know that*

$$E_{\mathcal{C}(Q)} + W_{\mathcal{C}(Q),t_0} > (1-\delta)2^{-(4+\epsilon)j(Q)},$$

then there exists $Q' \in \mathcal{C}(Q)$, so that

$$E_{Q'} + W_{Q',t_0} \geq 2^{-(4+\epsilon)j(Q')}.$$

Proof Let us assume the lemma is false, i.e.

$$E_{Q'} + W_{Q',t_0} < 2^{-(4+\epsilon)j(Q')}, \tag{8.3.3}$$

for all $Q' \in \mathcal{C}(Q)$.

On the other hand we have:

$$\begin{aligned} E_{\mathcal{C}(Q)} + W_{\mathcal{C}(Q),t_0} &= \sum_{Q' \in \mathcal{C}(Q)} [E_{Q'} + W_{Q',t_0}] \\ &= \sum_{k=1}^{\infty} \sum_{Q' \in \mathcal{C}^k(Q)} [E_{Q'} + W_{Q',t_0}]. \end{aligned} \tag{8.3.4}$$

Now because we are in dimension 3, there are exactly 2^{3k} elements $Q' \in \mathcal{C}^k(Q)$ with $j(Q') = j(Q)+k$. Thus by using (8.3.3) we can bound (8.3.4) from above by

$$\sum_{k=1}^{\infty} 2^{3k} \cdot 2^{-(4+\epsilon)(j(Q)+k)},$$

which is in turn the same as

$$2^{-(4+\epsilon)j(Q)} \sum_{k=1}^{\infty} 2^{-(\epsilon+1)k}. \tag{8.3.5}$$

Since

$$\sum_{k=1}^{\infty} 2^{-(\epsilon+1)k} < 1$$

we can choose $0 < \delta < 1$ such that

$$\sum_{k=1}^{\infty} 2^{-(\epsilon+1)k} = 1 - \delta.$$

Therefore (8.3.5) transforms into $(1-\delta)2^{-(4+\epsilon)j(Q)}$, and we obtain

$$E_{\mathcal{C}(Q)} + W_{\mathcal{C}(Q),t_0} < (1-\delta)2^{-(4+\epsilon)j(Q)},$$

which contradicts the assumption of our lemma. ■

Now we prove the main lemma.

Lemma 8.3.2.2 *Fix j_0 sufficiently large. Then there exists an ϵ, $0 < \epsilon < 1 - 4\alpha$ so that if at time t_0, we have*

$$E_Q + W_{Q,t_0} \geq 2^{-(4+\epsilon)j(Q)} \tag{8.3.6}$$

with $j(Q) > j_0$, then there is some t with $t < t_0 + T$, where

$$2^{\frac{(\epsilon-1)j(Q)}{2}} < T < 2^{-2\alpha j}$$

and a cube $Q' \in \mathcal{C}(Q)$ so that at time t, we have

$$E_{Q'} + W_{Q',t_0} \geq 2^{-(4+\epsilon)j(Q')}.$$

Proof We assume that the conclusion of the lemma is false, i.e.

$$E_{Q'} + W_{Q',t_0} < 2^{-(4+\epsilon)j(Q')}, \tag{8.3.7}$$

for all $Q' \in \mathcal{C}(Q)$, and for all $t \in [t_0, t_0 + T]$.

In light of Lemma 8.3.2.1 and Proposition 8.3.1.1, it must be the case that for all $t \in [t_0, t_0 + T]$, we have

$$E_Q(t) + W_{Q,t_0}(t) \geq \delta 2^{-(4+\epsilon)j(Q)}, \tag{8.3.8}$$

since otherwise because of the hypothesis (8.3.6) we would have

$$E_{\mathcal{C}(Q)} + W_{\mathcal{C}(Q)} > (1-\delta)2^{-(4+\epsilon)j(Q)}, \text{ at some } t \in [t_0, t_0 + T],$$

which would lead to a contradiction by Lemma 8.3.2.1.

Since $W_{Q,t_0}(t)$ is a monotone increasing function of t, (8.3.8) implies that

$$E_Q(t) + W_{Q,t_0}(t_0 + T) \geq \delta 2^{-(4+\epsilon)j(Q)}, \text{ for all } t \in [t_0, t_0 + T],$$

and therefore we have either

$$E_Q(t) \geq \frac{1}{2}\delta 2^{-(4+\epsilon)j(Q)}, \text{ for all } t \in [t_0, t_0 + T], \tag{8.3.9}$$

or

$$W_{Q,t_0}(t_0 + T) \geq \frac{1}{2}\delta 2^{-(4+\epsilon)j(Q)}. \tag{8.3.10}$$

We shall analyze those cases separately.

First let us assume (8.3.10). Let Q' an element of $\mathcal{C}^1(Q)$. Then $u_{Q'}$ satisfies

$$\frac{du_{Q'}}{dt} = 2^{\frac{5(j(Q)+1)}{2}} u_Q^2 - 2^{\frac{5(j(Q)+2)}{2}} u_{Q'} \Big(\sum_{Q'' \in \mathcal{C}^1(Q')} u_{Q''} \Big) - 2^{2\alpha(j(Q)+1)} u_{Q'}. \tag{8.3.11}$$

We shall integrate (8.3.11) on the time interval $[t_0, t_0+T]$.

In order to simplify our notation we introduce the following integrals:

$$\begin{aligned}
I_1 &:= \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+1)}{2}} u_Q^2, \\
I_2 &:= \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+2)}{2}} u_{Q'} \Big(\sum_{Q'' \in \mathcal{C}^1(Q')} u_{Q''} \Big), \\
I_3 &:= \int_{t_0}^{t_0+T} 2^{2\alpha(j(Q)+1)} u_{Q'}.
\end{aligned}$$

By using (8.3.10) we estimate I_1 and obtain

$$I_1 := \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+1)}{2}} u_Q^2 \gtrsim \delta 2^{-(\frac{3}{2}+2\alpha+\epsilon)j}. \tag{8.3.12}$$

We bound I_2 as follows:

$$\begin{aligned}
I_2 &:= \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+2)}{2}} u_{Q'} \Big(\sum_{Q'' \in \mathcal{C}^1(Q')} u_{Q''} \Big) \\
&\le \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+2)}{2}} u_{Q'} \Big| \sum_{Q'' \in \mathcal{C}^1(Q')} u_{Q''} \Big| \\
&\lesssim 2^{\frac{5j}{2}} \cdot T \cdot 2^{-(4+\epsilon)j} \cdot 2^3,
\end{aligned}$$

where the last inequality follows from (8.3.7). Thus

$$I_2 \lesssim T \cdot 2^{-(\frac{3}{2}+\epsilon)j}. \tag{8.3.13}$$

Similarly we use (8.3.7) in order to get

$$I_3 := \int_{t_0}^{t_0+T} 2^{2\alpha(j(Q)+1)} u_{Q'} \lesssim T \cdot 2^{(2\alpha-2-\frac{\epsilon}{2})j}. \tag{8.3.14}$$

One easily checks from (8.3.12), (8.3.13) and (8.3.14) that if $0 < \epsilon < 1 - 4\alpha$ and $T < 2^{-2\alpha j}$ then

$$I_1 \geq I_2,$$

as well as

$$I_1 \geq I_3.$$

Therefore after integrating (8.3.11) on the time interval $[t_0, t_0 + T]$ we conclude that

$$\int_{t_0}^{t_0+T} \frac{du_{Q'}}{dt} \gtrsim \delta 2^{-(\frac{3}{2}+2\alpha+\epsilon)j},$$

which is a contradiction by the fundamental theorem of calculus, since $\epsilon < 1 - 4\alpha$.

Now we are left to verify contradiction in the case that one has (8.3.9). Let us assume (8.3.9). Moreover since energy flows only in the direction of smaller cubes and is conserved up to waste baskets, it must be that for any child $Q' \in \mathcal{C}^1(Q)$, it is the case that

$$E_{\mathcal{C}(Q)} + W_{\mathcal{C}(Q),t_0} \geq \int_{t_0}^{t_0+T} E_{Q',in},$$

at time $t_0 + T$. Thus since

$$E_{Q',in} = 2^{\frac{5(j(Q)+1)}{2}} E_Q u_{Q'},$$

we must have

$$\int_{t_0}^{t_0+T} u_{Q'} \lesssim \frac{1-\delta}{\delta} 2^{\frac{-5j(Q)}{2}}. \tag{8.3.15}$$

We integrate (8.3.11) on the time interval $[t_0, t_0+T]$. By using (8.3.9) we bound I_1 as follows:

$$I_1 := \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+1)}{2}} u_Q^2 dt \gtrsim T \cdot \delta \cdot 2^{-(\frac{3}{2}+\epsilon)j}. \tag{8.3.16}$$

We bound I_2 as:

$$\begin{aligned} I_2 &:= \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+2)}{2}} u_{Q'} \Big(\sum_{Q'' \in \mathcal{C}^1(Q')} u_{Q''} \Big) \\ &\leq \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+2)}{2}} u_{Q'} \Big| \sum_{Q'' \in \mathcal{C}^1(Q')} u_{Q''} \Big| \\ &\lesssim \frac{1-\delta}{\delta} 2^{-(2+\frac{\epsilon}{2})j}, \end{aligned}$$

where the last inequality follows from (8.3.15) and (8.3.7). Thus

$$I_2 \lesssim \frac{1-\delta}{\delta} 2^{-(2+\frac{\epsilon}{2})j}. \tag{8.3.17}$$

By using (8.3.15) we obtain the following bound on I_3

$$I_3 := \int_{t_0}^{t_0+T} 2^{2\alpha(j(Q)+1)} u_{Q'} \lesssim 2^{(2\alpha-\frac{5}{2})j}. \tag{8.3.18}$$

Again one easily checks from (8.3.16), (8.3.17) and (8.3.18) that I_1 dominates provided that $0 < \epsilon < 1-4\alpha$ and $T > 2^{\frac{(\epsilon-1)j}{2}}$.

Therefore after integrating (8.3.11) on the time interval $[t_0, t_0+T]$ we conclude that

$$\int_{t_0}^{t_0+T} \frac{du_{Q'}}{dt} \gtrsim T \cdot \delta \cdot 2^{-(\frac{3}{2}+\epsilon)j},$$

which is a contradiction by the fundamental theorem of calculus, since $T > 2^{\frac{(\epsilon-1)j(Q)}{2}}$. ■

Lemma 8.3.2.3 *Fix j_0 sufficiently large. Then there exists an ϵ, $0 < \epsilon < 1-4\alpha$ such that if at time t_0 we have*

$$E_Q \gtrsim 2^{-(4+\epsilon)j(Q)} \tag{8.3.19}$$

with $j(Q) > j_0$, then there is some time t with $t < t_0 + T$, where

$$2^{\frac{(\epsilon-1)j(Q)}{2}} < T < 2^{-2\alpha j(Q)}$$

and a cube $Q' \in \mathcal{C}(Q)$ so that at time t, we have

$$E_{Q'} \gtrsim 2^{-(4+\epsilon)j(Q')}.$$

Proof We shall prove the lemma by contradiction. Assume the lemma is false, i.e.

$$E_{Q'} < 2^{-(4+\epsilon)j(Q')}, \text{ for all } Q' \in \mathcal{C}(Q), \text{ and for all } t \in [t_0, t_0 + T]. \tag{8.3.20}$$

However from the hypothesis (8.3.19) and Lemma 8.3.2.2 we can find time $t_1 < t_0 + T$, where

$$2^{\frac{(\epsilon-1)j(Q)}{2}} < T < 2^{-2\alpha j(Q)} \tag{8.3.21}$$

and a cube $Q_1 \in \mathcal{C}(Q)$ so that

$$E_{Q_1}(t_1) + W_{Q_1,t_0}(t_1) \geq 2^{-(4+\epsilon)j(Q_1)}. \tag{8.3.22}$$

On the other hand by using monotonicity of the function $W_{Q_1,t_0}(t)$ and (8.3.20) we calculate

$$\begin{aligned}
E_{Q_1}(t_1) + W_{Q_1,t_0}(t_1) &< E_{Q_1}(t_1) + W_{Q_1,t_0}(t_0+T) \\
&= E_{Q_1}(t_1) + \int_{t_0}^{t_0+T} 2^{2\alpha j(Q_1)} E_{Q_1} \\
&< 2^{-(4+\epsilon)j(Q_1)} + 2^{2\alpha j(Q_1)} \cdot 2^{-(4+\epsilon)j(Q_1)} \cdot T \\
&\lesssim 2^{-(4+\epsilon)j(Q_1)},
\end{aligned}$$

where the last inequality follows from (8.3.21).

Thus

$$E_{Q_1}(t_1) + W_{Q_1,t_0}(t_1) \lesssim 2^{-(4+\epsilon)j(Q_1)},$$

which contradicts (8.3.22), and the lemma is proved. ■

Corollary 8.3.3 *Let u be a solution to* (8.1.2) *which has all positive coefficients and at time 0, has $E_Q > 2^{-(4+\epsilon)j(Q)}$, for $j(Q) > j_0$ with j_0 as in the previous lemma. Then the $H^{2+\epsilon}$ norm of u becomes unbounded in finite time.*

Proof We apply the lemma 8.3.2.3. We find a cube Q_1 properly contained in Q and a time $t_1 < T < 2^{-2\alpha j(Q)}$ so that at t_1 we have $E_{Q_1} \gtrsim 2^{-(4+\epsilon)j(Q_1)}$.

We iterate this procedure finding a cube Q_k properly contained in Q_{k-1} and and a time t_k so that $t_{k-1} \leq t_k < t_{k-1} + T < t_{k-1} + 2^{-2\alpha j(Q_{k-1})}$ and at time t_k we have $E_{Q_k} \gtrsim 2^{-(4+\epsilon)j(Q_k)}$.

Estimating just using the coefficient at Q_k, we see that at time t_k, we have that

$$||u||_{H^{2+\epsilon}} \geq 2^{\frac{\epsilon j(Q_k)}{2}}.$$

Since $j(Q_k)$ is an increasing sequence of integers, this is going to ∞. However since,

$$t_k = (t_k - t_{k-1}) + (t_{k-1} - t_{k-2}) + \cdots + t_1 \leq 2^{-2\alpha j(Q)} + \sum_{l=1}^{k-1} 2^{-2\alpha j(Q_l)},$$

and the $j(Q_l)$'s are an increasing sequence of integers, we see that the sequence $\{t_k\}$ converges to a finite limit. ■

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- Analysis Seminar, Princeton University, May 2001
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