Lectures on gauge theory and symplectic geometry

TIM PERUTZ

1 What gauge theory has done for us

This lecture presents a classic example of the application of gauge theory to topology.

1.1 4-manifolds

The central *differential-geometric* feature of oriented 4-manifolds is the splitting of the 2-forms into self-dual (SD) and anti-self-dual (ASD) parts, the $+1$ and -1 eigenspaces of the Hodge star operator for a Riemannian metric:

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-.$$

The basic *topological* feature of closed, oriented 4-manifolds is the presence of a non-degenerate, symmetric bilinear form Q_X on $H^2(X;\mathbb{Z})/(\mathit{torsion})$,

$$(a,b) \mapsto (a \smile b)[X].$$

The signature of Q_X—the difference $\sigma(X) = b_+(X) - b_-(X)$ between the dimensions of maximal positive- and negative-definite subspaces—is called the signature of X. It is invariant under oriented cobordism.

Q_X has geometric meaning in homology, in de Rham cohomology and in Hodge cohomology. Poincaré duality identifies Q_X with the intersection form on $H_2(X;\mathbb{Z})/(\mathit{torsion})$, realized geometrically by intersection numbers of embedded oriented surfaces. Under the de Rham isomorphism $H^2_{dR}(X) \cong H^2(X;\mathbb{Z}) \otimes \mathbb{R}$, the $\mathbb{R}$-linear extension of the cup-product form corresponds to the wedge product form,

$$(\alpha,\beta) \mapsto \int_X \alpha \wedge \beta.$$

The g-harmonic 2-forms $\mathcal{H}^2(X)$ give a complete set of representatives for $H^2_{dR}(X)$. The splitting into SD/ASD parts commutes with the Hodge Laplacian and hence descends to harmonic forms:

$$\mathcal{H}^2 = \mathcal{H}^+ \oplus \mathcal{H}^-.$$

Here, $\mathcal{H}^2 := \Gamma(\Lambda^+) \cap \mathcal{H}^2$ is a positive-definite subspace for Q_X, maximal since it is complemented by $\mathcal{H}^2_-$ (defined similarly), a negative-definite subspace.

1.2 The instanton equation

If A is a connection in a principal bundle $P \to X$ over an oriented Riemannian 4-manifold, it has curvature

$$F_A \in \Omega^2(X; ad\, P) = \Gamma(\Lambda^2 T^*X \otimes ad\, P),$$

a 2-form valued in the adjoint bundle of Lie algebras. The curvature splits into self-dual and anti-self-dual parts:

$$F_A = F_A^+ + F_A^-,$$

sections of $\Lambda^\pm \otimes ad\, P$. An *instanton*, or ASD connection, is one such that

$$F_A^+ = 0.$$

The group of gauge transformations $\mathcal{G}_P = \mathrm{Aut}(P)$ acts on the connections by pullback, $u \cdot A = u^*A$ (on covariant derivatives, and thinking of G as a matrix group, $d_{u^*A} = d_A - u^{-2}(d_A u)u$). One has $F_{u^*A} = u^{-1}F_A u$, hence $F^+_{u^*A} = u^{-1}F^+_A u$.

When X is closed, the moduli space $M = M(X,P)$ of instantons modulo $\mathcal{G}_P$ is finite-dimensional by elliptic theory. Indeed, the *Coulomb gauge-fixing* equation

$$d^*_{A_0}(A - A_0) = 0,$$

contains a unique representative (up to constant gauge transformations) for each gauge orbit close to $[A_0]$ in a natural L^2 metric. The instanton and Coulomb equations jointly linearize to the operator

$$\delta_A = d_A^* + d_A^+ \colon \Omega^1 \to \Omega^0 \oplus \Omega^+.$$

This linear operator is elliptic, hence Fredholm (finite-dimensional kernel and cokernel). It has a certain index

$$\mathrm{ind}(\delta_A) = \dim\ker\delta_A - \dim\mathrm{coker}\,\delta_A.$$

If $M(X,P)$ is cut out transversely $[A]$, that is, δ_A is surjective, then $M(X,P)$ is locally modelled on $\ker\delta_A/\mathrm{stab}_{\mathcal{G}}(A)$, a quotient of a vector space of finite dimension $\mathrm{ind}\,\delta_A$ by the linear action of a group.

Example 1.1 Suppose that $b^+(X) = 0$. Then, for any metric, one has $\mathcal{H}^2 = \mathcal{H}^-$. Suppose that P is a principal $U(1)$-bundle. Let A_0 be a connection, ξ a 1-form, and $A = A_0 + i\xi$. Then $F_A = F_{A_0} + d\xi$. So we can represent any closed form in the class $[F_{A_0}] = -2\pi c_1(P)$ as F_A. In particular, we can choose A so that F_A is harmonic. Then $F_A^+ = 0$, so A is an instanton (called an *abelian instanton*, because $U(1)$ is abelian). If $b_1(X) = 0$, $[A]$ will be the unique gauge-orbit of instantons in P.

1.3 Donaldson's diagonalizability theorem

Donaldson theory extracts topological information about X from the moduli spaces of instantons. Without further ado, let's give a classic example, historically the first and still perhaps the most beautiful.

Theorem 1.2 (Donaldson) *Let X be a closed, oriented, simply connected 4-manifold with negative-definite intersection form Q_X. Then there is a basis for $H_2(X;\mathbb{Z})$ in which Q_X is represented by the matrix $-I$.*

Sketch proof. Fix a metric g on X, and an $SU(2)$-bundle $P \to X$. Let E be the associated $\mathbb{C}^2$-bundle, and choose P so that $c_2(E)[X] = 1$. We consider ASD connections A in P. Let M be the moduli space of ASD connections modulo the action of the group of gauge transformations $\mathcal{G} = \mathcal{G}_P$.

For each class $c \in H^2(X;\mathbb{Z})$ with $c^2[X] = -1$, we can construct an element of M as follows. Let $L \to X$ be a line bundle with $c_1(L) = c$. One has $E \cong L \oplus L^*$, since by the Whitney sum formula, $c_1(L \oplus L^*) = c - c = 0$ and $c_2(L \oplus L^*) = c_1(L)c_1(L^*) = -c^2 = 1$. We saw earlier that L carries a unique gauge-orbit of abelian instantons B. Under the isomorphism $E \cong L \oplus L^*$, the connection $B \oplus B^*$ is an ASD connection in E. The ASD connections we obtain this way are *reducible*, in that they are stabilized by a non-trivial group of constant gauge transformations (a copy of $U(1)$ in $\mathcal{G}$). We have constructed N_X of them, where N_X is the number of pairs $\pm c$ where $c \in H^2(X;\mathbb{Z})$ satisfies $c^2 = -1$.

One shows, using the hypothesis that $\pi_1(X) = \{1\}$, that the remaining ASD connections are *irreducible* (stabilized only by $\{\pm 1\} \subset \mathcal{G}$). For generic g, the subspace $M^* \subset M$ of irreducibles is a smooth, orientable manifold; its dimension is 5 (index theory). Near a reducible point, M is modeled on $\mathbb{C}^3/U(1)$ (action of $U(1) \subset \mathbb{C}$ by scalar multiplication), the cone on $\mathbb{C}P^2$.

For any $[A] \in M$, one has by Chern–Weil theory and the ASD equation,

$$\frac{1}{8\pi^2}\int_X |F_A|^2\, d\,vol = \frac{1}{8\pi^2}\int_X \operatorname{tr} F_A^2 = c_2(E)[X] = 1.$$

The space M is not compact, but Uhlenbeck understood that the only source of non-compactness is that the measure $|F_A|^2$ can concentrate in a very small ball B_ϵ. A very small ball is approximately isometric to standard $\mathbb{R}^4$, hence conformally approximately equivalent to $S^4 \setminus \{pt.\}$. The concentrated instantons approximate instantons in a bundle with $c_2 = 1$ over S^4, and in particular, have $\frac{1}{8\pi^2}\int_{B_\epsilon} |F_A|^2\, d\,vol \approx 1$. This means that concentration can occur near only *one* point. The concentrated instantons, those for which a proportion of at least $1 - \delta$ of the measure $\frac{1}{8\pi^2}|F_A|^2$ is concentrated in a ball of radius ϵ, form an open subset $M_{conc} \cong X \times (0,\delta)$ with compact complement. Donaldson proves that $M_{conc} \cong X \times (0,\delta)$ by a diffeomorphism that measures the center and degree of concentration of these instantons. In particular, M is non-empty even when $N_X = 0$.

By slicing off neighbourhoods of the singular points represented by the reducibles, one obtains from M a compact, oriented cobordism from X to a disjoint union of N copies of $\mathbb{C}P^2$ (with some orientations). From the cobordism-invariance of signature, one sees that $-b_2(X) = \sigma(X) = \sum_{j=1}^{N_X} \epsilon_j$, where $\epsilon_j = \pm 1$ is the signature of $\mathbb{C}P^2$ with one or other orientation. Hence $N_X \geq b_2(X)$. But now an easy algebraic lemma tells us that a negative-definite unimodular lattice of rank r, with at least r pairs $\pm c$ such that $c^2 = -1$, has a basis in which its matrix is $-I$. □

1.4 Notes

(1.1.) Basic information about 4-dimensional topology can be found in chapter 1 of Donaldson and Kronheimer's book [DK].

(1.2.) See [DK], chapter 2. The standard fact that a first-order elliptic operator D over a compact manifold is Fredholm can be proved as follows (cf. e.g. [Wel]). One shows that D satisfies estimates

$$\|u\|_{L^2_{k+1}} \leq C_k(\|Du\|_{L^2_k} + \|u\|_{L^2}),$$

where $\|u\|_{L^2_k} = \sum_{j=0}^k \|u^{(j)}\|_{L^2}$ and $\|u\|_{L^2}$ is the L^2 norm with respect to a Riemannian metric. Solutions of $Du = 0$ with $\|u\|_{L^2} = 1$ are therefore uniformly bounded in L^2_k for any k, hence also in C^k by the Sobolev embedding theorem. The Arzela–Ascoli theorem then implies that the L^2-unit ball in $\ker D$ is sequentially compact, hence that $\ker D$ is finite-dimensional. The same argument applies to the formal adjoint operator D^*, and $\ker D^* \cong (\operatorname{im} D)^\perp$, hence $\dim \operatorname{coker}(D) < \infty$.

(1.3). The proof of Donaldson's diagonalizability theorem is from [Don1]. There is an efficient proof in Seiberg–Witten theory, which does not require the hypothesis of simple connectivity [Nic].

References

[Don1] S.K. Donaldson, *An application of Yang–Mills theory to four-dimensional topology*, J. Differential Geom 18 (1983), no. 2, 279–315.

[DK] S. K. Donaldson, P. B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, Oxford University Press, 1990.

[Nic] L. Nicolaescu, *Notes on Seiberg–Witten theory*, Graduate Studies in Mathematics, 28, American Mathematical Society, 2000.

[Wel] R. O. Wells, *Differential analysis on complex manifolds*, 3 ed., Graduate Texts in Mathematics, 65, Springer, 2008.