Lectures on gauge theory and symplectic geometry

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2 The vortex equations and the Seiberg–Witten equations

2.1 Topological applications of gauge theory

Donaldson's instanton theory has largely, but not entirely, been superseded by *Seiberg–Witten* (SW) theory, which is based on a related equation that encodes similar topological information but usually in a more convenient way. SW invariants of 3- and 4-manifolds can be computed by cutting up the manifolds into simple pieces, then applying techniques from symplectic geometry, using Ozsváth–Szabó's machinery of Heegaard Floer theory.

The applications of Donaldson theory, SW theory and OS theory (that is, Heegaard Floer theory) are astonishingly wide-ranging. Here's an incomplete list of things that these theories illuminate:

(1) Non-existence of smooth structures on 4-dimensional homotopy-types (contrast Freedman's existence theorems for topological manifold structures).

(2) Non-diffeomorphism of 4-manifolds (contrast Freedman's homeomorphism theorems).

(3) Smooth topology of complex surfaces.

(4) Symplectic geometry in dimension 4, contact geometry in dimension 3:
   - Non-existence of symplectic structures.
   - Inequivalence of symplectic structures and of contact structures.
   - Existence of holomorphic curves and Reeb orbits.

(5) Certifying minimality of the genus of representatives of second homology classes in 3- and 4-manifolds.

(6) Uniqueness for surgery presentations of 3-manifolds; properties of knots.

In some of these areas, one or other of the theories is particularly effective. For the results on non-existence of smooth structures, SW theory gives markedly sharper results than Donaldson theory, essentially because the SW moduli space is a *compact, framed manifold*, defined up to framed cobordism, while the instanton moduli space is merely a manifold with a good compactification. An example is Furuta's theorem:

Theorem 2.1 (Furuta [Fur]) *Every closed, spin 4-manifold X satisfies* $b_2(X) \geq \frac{5}{4}|\sigma(X)|$.

SW theory is also stunningly effective in proving existence of holomorphic curves in symplectic 4-manifolds (Taubes) and Reeb orbits in contact 3-manifolds (Taubes); symplectic structures do not interact visibly with the ASD equations.

The most celebrated result about uniqueness of surgery presentation of 3-manifolds is the Property P theorem of Kronheimer–Mrowka [KM], which says that non-trivial surgery on a non-trivial knot in S^3 never results in a simply connected manifold. Here it is the instanton theory that it is successful, because it is concerned with the representations of the fundamental group (these are hard to see in SW and OS theory). The greatest success and novelty of the OS theory is perhaps in producing interesting knot invariants.

2.2 Limitations of gauge theory

(1) We have not classified simply connected smooth 4-manifolds.

(2) The Donaldson, SW and OS theories all have TQFT-type features, to be discussed in these lectures. But we do not know any axiomatic characterization of them.

(3) We have not found any significant interaction between gauge theory and hyperbolic geometry or geometrization of 3-manifolds. Gauge theory is effective in those problems about 3-manifolds that relate to 4-dimensional cobordisms.

(4) We have not integrated these theories with quantum Chern–Simons theory. (The Kapustin–Witten equations are a more relevant gauge theory for this purpose, but analytically they are not well-understood.)

2.3 Vortices

We'll approach the 4-dimensional Seiberg–Witten equations via their 2-dimensional antecedent (which is in fact a dimensionally-reduced version): the vortex equations. Let Σ be a closed, connected Riemannian surface. The metric g gives rise to a conformal (or complex) structure j and to an area form $\alpha = vol_g$. Let $L \to \Sigma$ be a hermitian line bundle of degree $d = c_1(L)[\Sigma]$. Consider pairs (A, ϕ), where A is a $U(1)$-connection in L, and ϕ is a C^∞ section of L. The vortex equations read

$$\bar{\partial}_A \phi = 0 \qquad \text{in } \Omega^{0,1}(\Sigma; L); \tag{1}$$

$$iF_A = (\tau - |\phi|^2)\alpha \qquad \text{in } \Omega^2(\Sigma). \tag{2}$$

Here $\tau \in \mathbb{R}$ is a parameter. Since $\int iF_A/2\pi = d$, there are solutions only when $\int \tau\alpha \geq d$ (moreover, when $\int \tau\alpha = d$, the only solutions have $\phi = 0$ and $F_A = 0$).

The gauge group $\mathcal{G} = C^\infty(\Sigma, U(1))$ operates on pairs (A, ϕ) by

$$u \cdot (A, \phi) = (u^* A, u\phi) = (A - u^{-1}du, u\phi),$$

preserving the vortices (that is, the solutions to the vortex equations). The moduli space $\mathit{Vor}(\Sigma, L)$ of gauge-equivalence classes of solutions which is naturally a complex manifold; the complex structure J acts on tangent vectors by $J(a, \psi) = (\star a, i\psi)$.

The operator $\overline{\partial}_A$ makes L a *holomorphic* line bundle (the holomorphic sections are those in the kernel of $\overline{\partial}_A$). The first equation says that L is a holomorphic line bundle and ϕ a holomorphic section. Hence, when $\int \tau\alpha > d$, one has a map

$$\mathit{Vor}(\Sigma, L) \to \left\{ (\mathcal{L}, \phi) \,\middle|\, \begin{array}{c} \mathcal{L} \text{ a holomorphic structure on } L \\ \phi \not\equiv 0 \text{ a holomorphic section} \end{array} \right\} / C^\infty(\Sigma, \mathbb{C}^*).$$

The complex moduli space on the right is better known as the symmetric product $\mathrm{Sym}^d(\Sigma) = \Sigma^{\times d}/S_d$, and the map is

$$v\colon \mathit{Vor}(\Sigma, L) \to \mathrm{Sym}^d(\Sigma), \quad [A, \phi] \mapsto \phi^{-1}(0).$$

Theorem 2.2 *The map v is biholomorphic.*

Remark When $d = 0$, one has a unique vortex, up to gauge (corresponding to $\mathrm{Sym}^0 \Sigma = \{\emptyset\}$). The connection A is flat of trivial holonomy (and so trivializes the bundle) and ϕ is constant.

Remark When $\tau \gg 0$, vortices (A, ϕ) 'localize' along their zero-sets. Let D_r be the union of discs of radius r centered at the points of $\phi^{-1}(0)$. One has $|F_A| \leq c \exp(-\tau^{1/2}\, \mathit{dist}(\cdot, \phi^{-1}(0)/c))$ with a constants c depending only on (Σ, L). Hence $|\phi|^2\alpha$ is exponentially close to $\tau\alpha$.

One can think of a vortex as smeared-out versions of a degree d divisor on Σ (and $1/\tau$ is the smearing parameter). A degree d divisor can be thought of as a point in the complex manifold $\mathrm{Sym}^d(\Sigma)$, or as a d-tuple of points on Σ.

Three points of view on vortices—as solutions to a PDE, as points in an algebro-geometric moduli space, or as tuples of points on the surface—and the relations between these viewpoints—are the prototypes for similar but more complicated constructions in 3 and 4 dimensions.

2.4 The Seiberg–Witten equations

We now work over a Riemannian 4-manifold X. The Seiberg–Witten equations read

$$D_A^+\phi = 0 \tag{3}$$

$$\rho(F_{A^t} + i\eta)^+ = (\phi^* \otimes \phi)_0. \tag{4}$$

We now explain the terms. We first fix a Spin^c-structure $\mathfrak{s}$. This is a choice from an $H^2(X;\mathbb{Z})$-torsor. We think of $\mathfrak{s}$ in differential-gometric terms as:

- A pair $\mathbb{S}^\pm \to X$ of hermitian 2-plane bundles, called the positive and negative spinor bundles.
- A bundle isomorphism $\rho\colon T^*X \otimes \mathbb{C} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{S}^+, \mathbb{S}^-)$, called Clifford multiplication, satisfying the relation that makes $\mathbb{S}^+_x \oplus \mathbb{S}^-_x$ a module for the Clifford algebra $\mathrm{Cliff}(T^*_xX)$:

$$\rho(f)^\dagger\rho(e) + \rho(e)^\dagger\rho(f) = -2g(e,f)\mathrm{id}_{\mathbb{S}^+}.$$

Define ρ on complex 2-forms by

$$\rho(e \wedge f) = \frac{1}{2}\left(\rho(e)^\dagger\rho(f) - \rho(f)^\dagger\rho(e)\right) \in \mathrm{End}(\mathbb{S}^+).$$

One checks that $\rho(\Lambda^+) = \mathfrak{su}(\mathbb{S}^+)$ and $\rho(\Lambda^-) = 0$.

In the SW equations, A is a connection in $\mathbb{S}^+$ which is compatible with Clifford multiplication, in that $d_A(\rho(\lambda)) = \rho(\nabla\lambda)$ when ∇ is built from the Levi-Civita connection and A. Such a connection induces a $U(1)$ connection A^t in $\Lambda^2\mathbb{S}^+$, and $A \mapsto A^t$ is a bijection between Clifford connections and $U(1)$-connections. One has $F_{A^t} \in i\Omega^2(X)$. In the equations, this curvature term appears alongside η, a closed 2-form. The equation is in $i\mathfrak{su}(\mathbb{S}^+)$, the trace-free hermitian endomorphisms. The other field ϕ is a section of $\mathbb{S}^+$, and $\phi^* \otimes \phi$ the resulting hermitian endomorphism. The symbol $(\cdot)_0$ means the tracefree part. Thus, if $\mathbb{S}^+_x = \mathbb{C}^2$, and $\phi = \alpha f_1 + \beta f_2$ in a local unitary frame (f_1, f_2), then

$$(\phi^* \otimes \phi)_0 = \begin{bmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\bar{\beta} \\ \bar{\alpha}\beta & \frac{1}{2}(|\beta|^2 - |\alpha^2|) \end{bmatrix}.$$

Finally, in the first of the two SW equations, $D_A^+ = \sum_j \rho(e_j)\nabla_{A,e_j}\colon \Gamma(\mathbb{S}^+) \to \Gamma(\mathbb{S}^-)$ is a Dirac operator.

We can also impose a global gauge-fixing equation, $d^*(A^t - A^t_0) = 0$. After doing so, the linearized Seiberg–Witten equations are elliptic. Discarding zeroth-order terms (which do not affect the symbol, nor the Fredholm index), the linearized equations read

$$(d^+ + d^*)(a) = 0, \quad D_A^+\psi = 0.$$

2.5 Notes and references

(2.1.) A good introduction to Seiberg–Witten theory is Morgan's book [Mor]; a terse but substantial survey is [Don2]. The foundational paper on the Ozsváth–Szabó theory is [OS].

(2.3) The vortex equations arose as a first-order Ansatz for the second-order Landau–Ginzburg model of electromagnetic fields in a superconducting magnet. The catch is that the Ansatz is applicable only when a certain physical parameter takes a special value, which in reality it does not. See [Wit] for an engaging account of this and related topics, and [JT] for a physically-motivated account of the mathematics of vortices. There are several proofs that the vortex moduli space is the symmetric product (the first is in [JT]); a conceptually attractive one is García-Prada's [Gar].

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