Lectures on gauge theory and symplectic geometry

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4 The monopole TQFT

4.1 The monopole equations in 3 dimensions

There is a version of the Seiberg–Witten equations for a Riemannian 3-manifold Y. A Spin^c-structure $\mathfrak{t}$ on Y consists of a rank 2 hermitian vector bundle $\mathbb{S} \to Y$ (the spinor bundle) and an oriented isometry $\rho\colon T^*Y \to \mathfrak{su}(\mathbb{S})$, where $\mathfrak{su}(\mathbb{S})$ has the metric $|a|^2 = -\operatorname{tr} a^2$. There's a notion of a Clifford connection B in $\mathbb{S}$, and each of these has an associated Dirac operator

$$D_B\colon \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$$

The Clifford multiplication ρ extends naturally to a map

$$\rho\colon \Lambda^* Y \otimes \mathbb{C} \to \mathrm{End}(\mathbb{S}).$$

The 3-dimensional Seiberg–Witten equations for a pair (B, ψ) of Clifford connection and spinor $\psi \in \Gamma(\mathbb{S})$ read

$$D_B\psi = 0, \quad \frac{1}{2}\rho(F_{B^t} - i\eta) - (\psi^* \otimes \psi)_0 = 0. \tag{1}$$

Here η is a chosen closed 2-form. We also impose a gauge-fixing condition

$$d^*(B^t - B^t_0) = 0.$$

The linearized equations define an operator $i\Omega^1 \oplus \Gamma(\mathbb{S}) \to i\Omega^0 \oplus i\Omega^1 \oplus \Gamma(\mathbb{S})$ which cannot possibly be elliptic (the symbol maps between vector spaces of different dimension, so can't be an isomorphism), but there's an easy fix (including an extra scalar field f with $df = 0$) that makes it elliptic.

4.2 Cylinders

On a cylinder $Y \times \mathbb{R}$, with translation-invariant metric, the Seiberg–Witten equations arise as a gradient flow equation. Fix a Spin^c-structure $\mathfrak{t}$ on Y, i.e., a rank 2 hermitian vector bundle $\mathbb{S} \to Y$ and an oriented isometry $\rho\colon T^*Y \to \mathfrak{su}(\mathbb{S})$. It extends naturally to a 4-dimensional Spin^c-structure on $Y \times \mathbb{R}$.

We consider the space $\mathcal{C}(Y, \mathfrak{t})$ of pairs (B, ψ) of Clifford connection B in $\mathbb{S}$ and $\psi \in \Gamma(\mathbb{S})$. There is a canonical bijection

$$C^\infty(\mathbb{R}, \mathcal{C}(Y, \mathfrak{t})) \leftrightarrow \{(A, \phi) \in \mathcal{C}(Y \times \mathbb{R}, \mathfrak{s}) : A \text{ in temporal gauge}\},$$

where temporal gauge means that A has vanishing dt-component. In this temporal gauge, solutions (A, ϕ) to the monopole equations are the same thing as solutions to the equation

$$\frac{d}{dt}(B(t), \psi(t)) + \nabla \mathcal{L}(B(t), \psi(t)) = 0$$

where $\mathcal{L}$ is the Chern–Simons–Dirac functional,

$$\mathcal{L}(B(t), \psi(t)) = -\frac{1}{8}\int_Y (iF_{B^t} + iF_{B_0^t} - \eta) \wedge (B^t - B_0^t) + \frac{1}{2}\int_Y \langle D_B\psi, \psi\rangle d\, vol_g.$$

The critical points of this functional—stationary solutions to the 4D monopole equations—are the 3D monopole equations. (The failure of ellipticity is related to temporal gauge; stationary solutions could have a constant but non-zero term $c\, dt$ in the connection B.)

$\mathcal{L}$ is invariant under the identity component of the gauge group. For a general gauge transformation $u\colon Y \to U(1)$, defining a cohomology class $[u] = u^*(dt)$, one has

$$\mathcal{L}(u \cdot (B, \psi)) - \mathcal{L}(B, \psi) = 2\pi^2 \int_Y [u] \wedge c_1(\mathbb{S}) \in 2\pi\mathbb{Z}.$$

4.3 Monopole Floer homology

The gradient flow interpretation of the Seiberg–Witten equations is highly suggestive:

- If X is a compact 4-manifold bounding Y, $\hat{X}$ its cylindrical completion, and η_X a closed extension of η to X, one should expect that monopoles on $\hat{X}$ (satisfying a 'finite energy' condition) will converge, in temporal gauge, to 3-dimensional η-monopoles on Y.
- One can expect to define an invariant $m(X, \eta_X) \in HM(Y, \eta)$, where $HM(Y, \eta)$, a 'monopole homology group', is the the 'elliptic Morse–Novikov' (or 'Floer') homology of the functional $\mathcal{L}_\eta\colon \mathcal{B}(Y) \to S^1$.

The first clause is correct. The second clause is also correct if $[\eta] \in H^2(Y; \mathbb{R})$ is chosen so as to forbid, for Chern–Weil reasons, the existence of reducible monopoles $(B, \psi = 0)$.

In general, it is too naive because of the complicating effect of reducible monopoles, which have non-trivial stabilizer $U(1) \subset \mathcal{G}_Y$.

Roughly, the corrected version—say when $[\eta] = 0$—goes as follows.

Let $\tilde{\mathcal{B}}(Y) = \mathcal{C}(Y)/\mathcal{G}_{Y,y}$ be the quotient of the configuration space $\mathcal{C}(Y)$ by the free action of the based gauge group $\mathcal{G}_{Y,y} = \{u \in \mathcal{G}_Y : u(y) = 1\}$. There is a residual action of $U(1)$ on $\tilde{\mathcal{B}}(Y)$, and we wish to consider $U(1)$-equivariant Morse homology of $\mathcal{L}$ on $P := \tilde{\mathcal{B}}$.

Inside P, we have the locus $Q = P^{fix}$ of $U(1)$-fixed points. We have the homotopy quotient (or Borel construction)

$$P_{U(1)} = P \times_{U(1)} S^\infty,$$

and inside it the subspace

$$Q_{U(1)} = Q \times_{U(1)} S^\infty \cong Q \times \mathbb{C}P^\infty.$$

We shall be interested in the long exact sequence for homology of the pair,

$$\cdots \to H_*(Q_{U(1)}) \to H_*(P_{U(1)}) \to H_*(Q_{U(1)}, P_{U(1)}) \to \ldots.$$

More accurately, we shall be interested in a long exact sequence

$$\to \overline{HM}_\bullet(Y) \to \check{HM}_\bullet(Y) \to \widehat{HM}_\bullet(Y) \to$$

of Morse–Floer homology groups for the functional $\mathcal{L}$ on P, constructed as 'semi-infinite' analogs of the homology groups of $Q_{U(1)}$, $P_{U(1)}$ and $(P_{U(1)}, Q_{U(1)})$.

These are the monopole Floer homology groups of Y. They are set up by Kronheimer–Mrowka in their book using a beautiful and unusual geometric model for $U(1)$-equivariant Morse–Floer homology of $\mathcal{L}$.

HM-bar. The least interesting of the groups is $\overline{HM}_\bullet(Y)$, which corresponds to $H_*(Q_{U(1)})$. It is constructed from solutions to a decoupled version of the monopole equations,

$$D_B\psi = \lambda\psi, \quad F_B = 0,$$

whose solutions are flat $U(1)$ connections B and Dirac eigenspinors ψ. This group is determined entirely by $H^1(Y;\mathbb{Z})$ with the triple cup-product form $\Lambda^3 H^1(Y;\mathbb{Z}) \to \mathbb{Z}$.

4.4 The TQFT

There is a cobordism category COB_{3+1}^{conn} whose objects are closed, oriented, *connected* 3-manifolds. A morphism $Y_1 \to Y_2$ is a diffeomorphism-class of compact, oriented cobordisms from Y_1 to Y_2. Seiberg–Witten theory extends to a functor defined on

COB^{conn}_{3+1}. It assigns to each object Y an exact triangle of abelian groups (actually, topological $\mathbb{Z}[[U]]$-modules)

$$\check{H}_\bullet \to \overline{H}_\bullet \to \widehat{H}_\bullet,$$

each graded by the set $\mathrm{Spin}^c(Y)$.

A cobordism X from Y_1 to Y_2 induces homomorphisms $HM(X)\colon HM_\bullet(Y_1) \to HM_\bullet(Y_2)$ on each version of monopole Floer theory, respecting the exact triangles. Composition of cobordisms corresponds to composition of homomorphisms.

Each Spin^c-structure on X gives rise to a map $HM(X, \mathfrak{s})$ between appropriate summands of the HM-groups, and $HM(X)$ is the sum of all of these.

If one wants to incorporate disconnected Y into the theory, a more elaborate algebraic construction will be needed, reflecting the structure of the cohomology of the ambient configuration space. I don't know how to do this.

One might expect that the SW invariant of a closed Spin^c 4-manifold X would be obtained by applying one of these homomorphisms $HM(X^0, \mathfrak{s})$ to the cobordism X^0 from S^3 to S^3 obtained by puncturing X twice. Note, however, that the maps in the monopole Floer theory exist regardless of $b^+(X)$. And in fact the map $HM(X^0, \mathfrak{s})$ is zero in all three theories.

The SW invariant of X is extracted from the Floer theory by a 'secondary cohomology operation', secondary in the sense that it is only well-defined when $b^+(X) > 0$ (and is multi-valued if $b^+(X) = 1$). When $b^+(X) > 0$, there are no *reducible* solutions to the monopole equations. The map $\overline{HM}(X, \mathfrak{s})$, which counts only reducible solutions, is zero because its defining moduli spaces are empty. This emptiness facilitates the construction of a diagonal map

$$\check{HM}(X, s) \to \widehat{HM}(X, s)$$

factoring $\widehat{HM}(X, s)$. It is this diagonal map from which the Seiberg–Witten invariant can be extracted.