# Lectures on gauge theory and symplectic geometry

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## 5 Monopoles and Lagrangian submanifolds

### 5.1 Monopoles on 3-manifolds with boundary

Suppose $Y$ is an oriented, Riemannian 3-manifold with boundary $\partial Y =: \Sigma$. Let $\mathfrak{t}$ be a $\mathrm{Spin}^c$-structure and $\eta$ a closed 2-form on $Y$. We have the configuration space $\mathcal{C}(Y, \mathfrak{t})$, and the monopole moduli space $\mathcal{L} = \mathcal{L}(g, \eta)$ inside it.

*Question*: What sort of structure does $\mathcal{L}$ have?

*Answer*: there's a restriction map $r\colon \mathcal{L} \to V := \mathcal{C}(Y|_\Sigma, \mathfrak{t}|_\Sigma)$, intertwining the actions of the gauge groups $\mathcal{G}_Y$ and $\mathcal{G}_\Sigma$. The (affine) vector space $V$ has a canonical symplectic structure. If $\eta$ is chosen to that $2\pi c_1(\mathbb{S}) - [\eta] \neq 0 \in H^2(Y;\mathbb{R})$—so that there are no reducible monopoles—then $r$ is a Lagrangian embedding.

(Of course, $V$ is infinite-dimensional. This is still true after passing to $\mathcal{G}_Y$-orbits—equivalently, passing to $U(1)$-orbits on the Coulomb gauge slice.)

Now let $\hat{Y} = Y \cup_\Sigma (\Sigma \times \mathbb{R}_+)$ be the cylindrical completion of $Y$. Here we assume that the metric $g$ is a product in a collar neighbourhood of $\Sigma \subset Y$, and we extend this product form over the cylindrical end. Extend $\eta$ to $\hat{\eta}$ on $\hat{Y}$ in the obvious way, and consider the moduli space $\hat{\mathcal{L}}$ of *finite-energy* $\hat{\eta}$-monopoles on $\hat{Y}$.

*Variant question*: What structure does $\hat{\mathcal{L}}$ have?

*Answer* : Suppose that $\mathbb{S}|_\Sigma = L \oplus (K^{-1}L)$, and say $\eta|_\Sigma = \tau vol$ where $\tau > 2\pi\,\mathrm{area}(\Sigma)$. The vortex moduli space $\mathrm{Vor}(L,\tau)$ is the symplectic reduction of $\mathcal{C}(Y|_\Sigma, \mathfrak{t})$ by the Hamiltonian action of $\mathcal{G}_\Sigma$ at the moment-map value $\tau\, vol$. There is a natural map

$$a_Y\colon \hat{\mathcal{L}}/\mathcal{G}_Y \to \mathrm{Vor}(L,\tau)$$

taking a monopole to its asymptotic limit on the cylindrical end, and this is a Lagrangian immersion.

Note that the vortex moduli space is finite-dimensional (it is $\mathrm{Sym}^{\deg L}\Sigma$).

These answers are well in line with general elliptic theory, but the proofs are substantial. They are due to T. Nguyen (PhD thesis, MIT, 2011).

I claim that, in extensions of monopole Floer theory which encompass 3-manifolds with boundary, the Lagrangian immersions $a_Y$ should be considered as the basic objects.

## 5.2 Elliptic theory and Lagrangian subspaces

Consider a compact, oriented Riemannian manifold $Z$ with boundary, and a first-order elliptic operator $D$ acting in sections of a vector bundle $E \to Y$. We assume that $D$ is formally self-adjoint, i.e.,

$$\int_Z \langle u, Dv\rangle d\,vol = \int_Z \langle Du, v\rangle d\,vol$$

for all $u, v \in \Gamma(E)$ such that $u$ is supported compactly in $\mathrm{int}(Z)$.

When $u$ does not vanish at the boundary, one has 'Green's formula'

$$\int_Z (\langle u, Dv\rangle - \langle Du, v\rangle) d\,vol = i \int_{\partial Z} \langle \sigma(o) r(u), r(v)\rangle d\,vol,$$

wherer $r$ denotes restriction to $\partial Z$, $o$ is the outward unit conormal, and $\sigma$ the symbol of $D$. Let

$$\omega(f, g) = i \int_{\partial Z} \langle \sigma(o) f, g\rangle d\,vol, \quad f, g \in \Gamma(V|_{\partial Z}).$$

Then $\omega$ is a symplectic pairing, and evidently $(\ker D)|_Z$ is an isotropic subspace.

A much harder fact is that $(\ker D)|_Z$ is actually a *Lagrangian* subspace [?]. The point is that an isotropic complement can be constructed, roughly speaking, by doubling $Z$ to a closed manifold $\mathrm{D}Z = Z \cup (-Z)$, doubling $D$ to a self-adjoint elliptic operator $\tilde{D}$ over $X$, and considering the image over $\partial Z$ of $\ker(\tilde{D}|_{-Z}))$.)

**Example 5.1** Consider the Hodge operator $d + d^*$ on $\Omega^*_Z$. Its kernel—the harmonic forms—define a Lagrangian subspace of $\Omega^*_{\partial Z} \oplus \Omega^*_{\partial Z}$.

**Example 5.2** On a compact 3-manifold $Y$, the flat connections in a line bundle $L \to Y$ restrict to a Lagrangian subspace of the flat connections in $L|_{\partial Y}$. Here the symplectic structure is $(a, b) \mapsto \int_{\partial Y} a \wedge b$.

The linearized 3-dimensional monopole equations fall into this framework, and this explains why monopoles restrict to give a Lagrangian submanifold of the configuration space of the boundary.

Note that $\mathcal{L}$ embeds into $\mathcal{C}(\partial Y)$ by a unique continuation principle for solutions to the Dirac equation $D_B \psi = 0$.

The statement about immersions into vortex space requires additional analysis.

## 5.3 Monopole Floer homology versus Heegaard Floer homology

Monopole Floer homology is isomorphic to Heegaard Floer homology, as has been proved by Kutluhan–Lee–Taubes. Part of the statement is that

$$\widecheck{HM}_\bullet(Y) \cong HF^+(Y),$$

and I shall concentrate on this aspect. However, most elements of the proof I shall describe are imaginary: they have not been carried out.

We consider a Morse function $f$ on the closed 3-manifold $Y$, self-indexing with one minimum $x_-$ and one maximum $x_+$.

**Step 1: puncturing $Y$.** Let $Y^{\circ\circ} = Y \setminus \{x_-, x_+\}$. We can choose a metric $g$ on $Y^{\circ\circ}$ which is cylindrical near the punctures, and (cf. Calabi's work) such that $f$ is harmonic: $d^*df = 0$. Thus $\eta := \star df$ is a closed 2-form. We consider Spin$^c$-structures on $Y^{\circ\circ}$ whose spinor bundle $\mathbb{S}$ restricts to the linking 2-spheres of the punctures as $\mathbf{1} \oplus K_{S^2}^{-1}$. All finite-energy $\tau\eta$-monopoles on $Y^{\circ\circ}$ asymptote (up to gauge) to the canonical vortex in $Vor(L, \tau) = \{pt.\}$. Using these monopoles, and their 4-dimensional analogues on $Y^{\circ\circ} \times \mathbb{R}$, one should be able to form a version of Floer homology

$$HM_\bullet(Y^\circ, \tau\eta).$$

The coefficients are taken to be in the ring $R = \mathbb{Z}[[U]]$.

After choosing a path in $Y$ from $x_-$ to $x_+$ (e.g. a gradient flowline), one can identify Spin$^c$-structures on $Y^{\circ\circ}$ which are canonical on the ends with Spin$^c$-structures on $Y$. Using this identification, one should have $HM_\bullet(Y^{\circ\circ}, \tau\eta) \cong \widecheck{HM}_\bullet(Y)$.

This seems well within the range of current methods, and I will not comment further except to point out that $HM_\bullet(Y^{\circ\circ}, \tau\eta)$ involves only *irreducible* monopoles.

**Step 2: the Atiyah–Floer isomorphism.** $f$ divides $Y^{\circ\circ}$ into two punctured handlebodies, $U_\alpha^\circ$ and $U_\beta^\circ$, meeting along a surface $\Sigma$. Their cylindrical completions give rise to immersed Lagrangian submanifolds

$$\mathbb{L}_\alpha, \mathbb{L}_\beta \subset \mathrm{Vor}(E, \tau),$$

where $E \to \Sigma$ is a line bundle of degree $g$, the genus of $\Sigma$. I claim that these Lagrangians are actually embedded. (At the level of reducible monopoles, this amounts to the fact that $H^1(U_\alpha^\circ) \to H^1(\Sigma)$ is injective, ditto for $U_\beta^\circ$.) Moreover, they are tori (one way to approach proving this is to look at how they degenerate as $\tau$ decreases to the parameter at which all the vortices and monopoles are reducible).

The goal of this step is to prove a version of the Atiyah–Floer conjecture,

$$HF(\mathbb{L}_\alpha, \mathbb{L}_\beta) \cong HM_\bullet(Y^{\circ\circ}, \tau\eta).$$

There is indeed a bijection between the generators for the respective complexes, arising from the gluing theory of monopoles. Relating the differentials is very considerably more challenging. Proving such a statement will be key in extending monopole Floer homology to 2 dimensions.

**Step 3: the Taubes limit.** Now we take the limit $\tau \to \infty$. Note that $Vor(E,\tau) \cong \mathrm{Sym}^g(\Sigma)$ canonically. This isomorphism induces Kähler forms $\omega_\tau$ in varying cohomology classes. There are two things to prove. One is that the modules $HF(\mathbb{L}_{\alpha,\tau}, \mathbb{L}_{\beta,\tau})$ are independent of $\tau$. If all these groups are set up to have identical formal properties, this appears to be a reasonable (though not at all trivial) instance of the continuity of Floer homology. Second, one should prove that in the limit $\tau \to \infty$, $\mathbb{L}_{\alpha,\tau}$ is smoothly isotoped to the Heegaard torus $\mathbb{T}_\beta$ (similarly for $\beta$). Indeed, in this limit, monopoles on the cylindical completion of $U_\alpha^\circ$ localize along gradient flowlines for $f$, with these flowlines appearing as the (limiting) zero-sets of $\alpha$-spinors.

Note that in this approach, the different versions of the homology theories ($HF^\pm$ and $HF^\infty$ on the Heegaard side) should arise in a fairly simple way by taking the coefficient rings to be different versions of the Novikov ring.