RESEARCH STATEMENT

PEDRO FERNANDO MORALES ALMAZÁN

Vision

I deeply believe a fundamental part of mathematical research is the ability to do connections within different areas of mathematics. Many of the most celebrated results and major breakthroughs in mathematics arose from realizing these type of connections among previously unrelated areas. I think that a key factor to propose and develop mathematical research is to seek for these connections and to look at problems in various points of view. As in the language of differential geometry one could say, *sometimes changing the local coordinates can help simplify the understanding of the problem.* Part of the intertwine among different areas of mathematics includes networking and scientific sharing among mathematicians. Personally, I think about mathematics as a social activity. Collaboration constitutes a fundamental pillar in the research process. Nowadays, specialization in mathematics is been taken to the highest level, therefore interdisciplinary sharing among mathematicians is of great importance and opens new directions in research development.

Mathematics is in itself a language that has evolved through time. Specialization in mathematics has become even more evident, making mathematicians from different fields to speak different dialects. Part of great achievement in research then is made by understanding these dialects and to know how to share within people from different study areas within mathematics.

Introduction

My current research focuses on spectral zeta functions and their applications to regularization problems in Quantum Field Theory and Number Theory. Specifically I have been studying their relationship with quantum field effects, with a particular emphasis on the Casimir effect, and regularization of divergent series appearing in Number Theory.

My work has been focused on finding the analytic continuation of spectral zeta functions that arise from studying differential operators defined on compact manifolds subject to different boundary conditions. This methodology provides an efficient way of suppressing the presence of undefined series and infinities that often come up in the study of quantum field theories.

I am also interested in finding connections between the geometry of manifolds, boundary conditions, and their influence in the resulting spectral zeta function. Studying the impact of these characteristics on the eigenvalues and their distribution is essential to achieve a better understanding of spectral functions and their physical implications.

Background

The Casimir effect is a quantum field effect result of the zero-point energy fluctuations in a quantum mechanical system [29, 6]. Given a Hamiltonian H, the eigenvalue problem associated with the system,

$$H\Psi = \lambda\Psi \tag{1}$$

gives the fundamental frequencies at which the system resonates. Quantizing the harmonic oscillator shows that the zero-point energy is given by

$$E = \frac{\hbar\lambda}{2} \tag{2}$$

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for λ, the fundamental frequency of the oscillator. Therefore, it follows the formal definition of the Casimir energy

$$E = \frac{\hbar}{2} \sum_{\lambda \in \sigma(H)} \lambda \tag{3}$$

for $\sigma(H)$ the set of eigenvalues of H [29, 21].

In a physical context, most of the Hamiltonians arise as differential operators acting on the space $L^2(M)$ of finite energy wave functions defined on a manifold. In order to have a consistent theory, it is essential to have a self-adjoint differential operator that depends on the geometry of the manifold and the boundary conditions. Given H to be a self-adjoint operator acting on a Hilbert space, there is a characterization of its spectrum given by [8, 35],

Theorem 1 (Spectrum Self-Adjoint Operator)**.** *If H is a self-adjoint operator acting on a Hilbert space, then its point spectrum is real, discrete and with no limit point besides infinity.*

This result states that it is possible to label the eigenvalues as $\{\lambda_n\}_{n=1}^{\infty}$ and they are such that $\lambda_1 \leq \lambda_2 \leq \dots \lambda_n \leq \dots$ and $\lambda_n \to \infty$ as $n \to \infty$. Because of this, Equation (3) is not going to be well defined, as the expression diverges. In order to avoid such inconvenience, it is common [29, 21, 37, 19, 10] to use the help of the spectral zeta functions defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \lambda_n^{-2s} \, , \tag{4}$$

where the λ_n are taken to be distinct from zero. We have that this expression will converge for sufficiently large $\Re(s)$ [41, 21, 10, 37]. Then, it is possible to find the Casimir energy by considering $s = -1/2$ on the spectral zeta function. As $s = -1/2$ is not in the region of convergence of ζ, complex analysis techniques are used to find a meromorphic continuation of the spectral function to a region containing $s = -1/2$. This technique is known as Zeta Function Regularization, which is widely used to regularize divergent expressions arising from physical systems [29, 21, 37, 19, 10].

Traditionally, regularizing expressions using the zeta function scheme requires explicit knowledge of the eigenvalues of the operator. However, finding the actual eigenvalues for a configuration is, in general, not possible. Hence, one cannot directly calculate the spectral zeta function nor its analytic continuation.

Recently, a method has been developed to overcome this problem, to find the zeta function and its analytic continuation without explicitly knowing the eigenvalues [21, 23]. To achieve this purpose, one can make use of *Cauchy's Residue Theorem*, which states that [38],

Theorem 2 (Cauchy's Residue Theorem)**.** *Given F a meromorphic function and a close curve γ, then*

$$\frac{1}{2\pi i} \int_{\gamma} F(z) dz = \sum_{a \in A} I(\gamma, a) Res\, F(z)|_{z=a}$$

where A is the set of poles of F inside γ, and I is the winding number of γ around a.

By finding a suitable function F whose zeros are the eigenvalues of H, and by applying these previous results, it is possible to find an expression for the zeta function. The characteristic equation that the eigenvalues satisfy can be obtained by imposing the boundary conditions of the problem. If $f(\lambda)$ is such that $f(\lambda) = 0$ gives the eigenvalues of H, then one can consider the function

$$F(\lambda) = \lambda^{-2s} \frac{f'(\lambda)}{f(\lambda)} \tag{5}$$

which is meromorphic in the entire complex plane and has poles at the eigenvalues of H with

$$\text{Res}\, F(\lambda)|_{\lambda=\lambda_n} = \lambda_n^{-2s} \, . \tag{6}$$

In order to apply Cauchy's Residue Theorem to this function and capture all of the eigenvalues some regularity conditions of F at infinity are required, as an infinite contour must be considered. This provides an integral representation for the zeta function, which is well suited for performing its analytic continuation.

By subtracting asymptotic terms of F as $\lambda \to \infty$, it is possible to extend the convergence region of the integral representation of the zeta function further to the left. This procedure can be continued to include any point in the complex plane, in particular to a region containing $s = -1/2$ [21, 23, 10].

Calculations of the spectral zeta function are hence deeply connected to the geometry of the underlying manifold and the boundary conditions applied, although a direct method to describe such connections is still elusive at this point[16, 2].

This method involving an integral representation and its analytic continuation of the spectral zeta function provides a path to investigate other properties of these differential operators, such as the determinant of the operator, which gives information about the one loop contribution of the effective action [21, 19, 42, 18].

Moreover, other important spectral functions, such as the heat kernel, can be related to the spectral zeta function in a geometric way. Information regarding geometric properties of the underlying space such as the heat kernel coefficients can be related with the poles of the spectral zeta function associated to the space. Hence this method provides a very efficient way of finding such information without the need of appealing to other functions[21, 29, 2].

Other type of information encoded in the spectral zeta function relates to the cohomology of the ambient manifold. The de Rham cohomology as well as the Reidemeister torsion are examples of such information also encoded in the spectral zeta function [34].

Current and Past Research

In order to have a well defined physical quantity, it is presently common to work with a configuration called a *Piston* [11, 21, 14, 13]. Basically, such a configuration is the result of joining two configurations by a common boundary and boundary conditions, such that this boundary is movable in a normal direction. By finding the resulting spectral zeta function of this new configuration, it is possible to find the self-energy of the system without having undefined quantities or infinities. By calculating the derivative of the spectral zeta function with respect to the position of the piston, one is able to find the effective force acting on the piston, and hence, determine whether there is a definite behavior for the piston as it is attracted or repelled to the closest wall.

The research that has been made in this area is still connected to the geometry of the manifold and to the boundary conditions in a unknown manner [21, 13, 14, 32], resulting in completely different behaviors for the piston when the boundary conditions are perturbed. Because of this reason, currently, the behavior of the system must currently be studied separately for each different configuration.

Classical calculations are made in the context of Dirichlet or Neumann boundary conditions at the piston, but a more general approach has recently been made that considers potentials of different properties as opposed to the usual ones [12, 14, 13, 32]. In pursuing this approach, I have been working with the usual configuration of parallel plates, with a piston modeled by a delta potential. By using the method described above, I found a characteristic equation satisfied by the eigenvalues of the differential operator and with this, defined an integral representation of the associated zeta function. This representation together with the asymptotic expansion of the secular equation helped me to find an analytic continuation of the zeta function in order to include $s = 0, -1/2$ into the domain of convergence [32],

Theorem 3 (Delta Potential)**.** *Let $M = [0, L] \times N$ be an m dimensional manifold, where N is a smooth Riemannian manifold possibly with boundary. Consider the differential operator*

$$P = -\frac{\partial^2}{\partial x^2} - \Delta_N + \sigma\delta(x-a) \tag{7}$$

where $\sigma > 0$ and $0 < a < L$, defined on M subject to Dirichlet boundary conditions. Then the zeta function associated with P is given by

$$\zeta_P(s) = \zeta_f(s) + \zeta_{as}(s) \tag{8}$$

where

$$\zeta_f(s) = \frac{\sin \pi s}{\pi} \sum_{\ell} \int_{\eta_\ell}^{\infty} dk\,(k^2 - \eta_\ell^2)^{-s} \frac{d}{dk} \left[\ln F(ik) - kL + \ln(2k) - \sum_{j=1}^{m} (-1)^{j+1} \left(\frac{\sigma}{2k}\right)^j \frac{1}{j} \right] \tag{9}$$

and

$$\zeta_{as}(s) = \frac{L\Gamma\left(s - \frac{1}{2}\right)}{2\sqrt{\pi}\Gamma(s)} \zeta_N\left(s - \frac{1}{2}\right) - \frac{1}{2}\zeta_N(s) + \sum_{j=1}^{m} (-1)^j \left(\frac{\sigma}{2}\right)^j \frac{\Gamma\left(\frac{j}{2} + s\right)}{\Gamma\left(1 + \frac{j}{2}\right)\Gamma(s)} \zeta_N\left(s + \frac{j}{2}\right) \tag{10}$$

where η_ℓ are the eigenvalues of the laplacian on N and F is the secular equation satisfied by the eigenvalues of P and is given by

$$F(\nu) = \frac{1}{\nu^2}\left(\sigma \sin(\nu a) \sin(\nu[L - a]) + \nu \sin(\nu L)\right) \tag{11}$$

With this, I was able to calculate the functional determinant for the differential operator P and to analytically determine sign of the associated Casimir force, and hence the behavior of the piston as to be attracted to the closest wall.

Following the same train of thought, I also worked the case where the piston is modeled by a rectangular potential, and also found a definite result in the force acting on the piston. Definite results for the force are uncommon due to the complexity of the spectral zeta functions usually obtained, and often have to be evaluated by numerical methods.

In an attempt to find a more general result, I worked on a project involving a general smooth potential with compact support [12, 3]. The initial approach was the same as the previous cases, however as working with an unknown potential, direct asymptotic expansion can be found[12, 28]. In order to go around this problem I used the WKB approximation to have a recursive algorithm for calculating the asymptotic orders of the secular equation.[12, 3]

Theorem 4 (Compact Supported Potentials)**.** *Let $M = [0, L] \times N$ be an m dimensional manifold, where N is a smooth Riemannian manifold possibly with boundary. Consider the differential operator*

$$P = -\frac{\partial^2}{\partial x^2} - \Delta_N + V(x) \tag{12}$$

where $V(x)$ is supported on $[0, L]$, defined on M subject to Dirichlet boundary conditions. Then the zeta function associated with P is given by

$$\zeta_P(s) = \zeta_f(s) + \zeta_{as}(s) \tag{13}$$

where

$$\zeta_f(s) = \frac{\sin \pi s}{\pi} \sum_{\ell} \int_{\eta_\ell}^{\infty} dk\,(k^2 - \eta_\ell^2)^{-s} \frac{d}{dk} \left[\ln u_{ik}(L) - kL + \ln(2k) - \sum_{j=1}^{m} d_j k^{-j} \right] \tag{14}$$

and

$$\zeta_{as}(s) = \frac{L\Gamma\left(s - \frac{1}{2}\right)}{\sqrt{\pi}\Gamma(s)} \zeta_N\left(s - \frac{1}{2}\right) - \zeta_N(s) - \sum_{j=1}^{m} j d_j \frac{\Gamma\left(\frac{j}{2} + s\right)}{\Gamma\left(1 + \frac{j}{2}\right)\Gamma(s)} \zeta_N\left(s + \frac{j}{2}\right) \tag{15}$$

where η_ℓ are the eigenvalues of the laplacian on N and u_{ik} are solutions of the ODE

$$-u''(x) + (V(x) + k^2)u(x) = 0 \tag{16}$$

with Dirichlet boundary conditions, and d_j can be found recursively by

$$d_j = \int_0^L S_j(x)dx \tag{17}$$

and

$$S_{-1}(x) = 1, \quad S_0(x) = 0, \quad S_1(x) = \frac{1}{2}V(x) \tag{18}$$

$$S_{i+1}(x) = -\frac{1}{2}\left(S_i'(x) + \sum_{j=0}^{i} S_j(x)S_{i-j}(x)\right) \tag{19}$$

This method has the advantage that is well suited for numerical calculations and it is independent of the actual potential, as long as it is smooth. Understanding this problem has led to results involving the universal coefficients of the heat kernel expansion in terms of the geometric properties of the underlying manifold [3, 12].

I also worked on a generalization of this work into a higher dimensional setting [31]. Studying higher dimensional spherical pistons with spherically symmetric smooth potentials is the next natural step to understand better the behavior of the Casimir effect and it dependence on the geometric structure of the system analyzed.

Another project was to analyze the effect on small surface perturbations on the change in the Casimir energy [30]. The Casimir is greatly influenced by the local geometry [21, 29, 7]. One of the most interesting questions in the field is to analyze what is the dependence on the geometry and boundary conditions. By analyzing perturbations on surfaces of revolution, I seek to obtain better understanding of these implications. The understanding of the geometric dependence of the Casimir effect is of great importance nowadays, as nano technology is getting closer and closer, and a deeper understanding of the quantum world is imminent for the development of new technologies. This study can clarify the geometric preference for quantum systems for a specific type of curvature and boundary conditions.

Currently I am working on generalizing my results on perturbed surfaces of revolution to higher dimensions by considering warped manifolds $M = I \times_f N$, where I is an interval, N is a compact Riemannian manifold, and f is the warping function. Here I consider the effects of a perturbation $f \mapsto f + \epsilon g$ on the zeta function associated with the Laplacian on M. This shows to provide well defined quantities for even dimension manifolds but to have a different behavior in odd dimensions.

In order to better understand the behavior contour integrals appearing in integral representations of zeta functions, I am also studying the effects of taking contour integrals of one forms over an essential singularity. This is important as many zeta functions despite not having a pole at infinity will present an essential singularity at infinity that will contribute when computing contour integrals over paths that go to infinity, specially over vertical lines. These integrals will present a Stokes-phenomenon-like behavior, in which there are different results depending the angle in which the contour passes through the essential singularity.

Future Plans

The study of spectral functions continues to gain increased appreciation throughout various branches of mathematics and physics. Because of the sheer amount of geometric information these functions encode, this field not only provides an attractive area of study, but also a potentially fertile area of work. Currently, many connections between such functions, the spectrum of differential operators, and the geometry of the underlying space together with the boundary conditions remain to be discovered and explored.

As is presently known, there can be spaces with different geometric structures that give rise to identical spectral properties. The existence of non-isometric isospectral manifolds provided a starting point to investigate which of the geometric properties contribute to the behavior of the spectrum of these operators [16].

One of the goals of the field is to provide a way to analyze the behavior of the spectral functions, by a direct characterization in terms of geometric properties of the domain and the boundary properties imposed.

My research has been directed towards finding a general way of finding spectral zeta functions independently of the boundary conditions and the geometry of the space. I have been doing this by studying broader classes of potentials in the setup of Casimir pistons.

My research plans cover the further study of the effect of perturbations on zeta functions. Not only perturbations involving the manifold but also the effect of perturbing boundary conditions. By this I seek to discover the main factors that influence the behavior of zeta functions as well as the change at special values, for example the Casimir energy at $s = -1/2$ and the functional determinant at $s = 0$.

I plan to study the effect of transformations in the metric on the zeta function and its special values. Using conformal transformations is possible to identify invariants in the structure of zeta functions and in the calculation of special values. In the case of the Casimir energy, a reverse approach can also be made. By seeking to minimize the Casimir force acting on a space, one can define a Casimir flow that changes the metric on the space in order to reduce the force.

The study of the asymptotic behavior of zeta functions and functions defining them is of prime importance. I plan to continue my research on the effect of essential singularities on contour integrals that appear in integral representations of zeta functions. This study will also provide tools to handle divergent series appearing in number theory in order to regularize them.

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Department of Mathematics, The University of Texas at Austin, 2515 Speedway, Austin, TX 78712
E-mail address: pmorales@math.utexas.edu