M375T/M396C: Topics in Complex Networks **Spring 2013**

Lecture 7 — 02/07/2013

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7.1 The Configuration Model

At the end of the previous lecture we discussed the Configuration model and saw that it has degree distribution $d(k) \sim \text{Bin}(n-1,p)$ where $n \in \mathbb{N}$. We also observed that as we let $n \to \infty$, $\text{Bin}(n-1,p) \to \text{Poisson}(\lambda)$ for some parameter λ. We use this model to construct a random graph with fixed degree distribution and do this with probability $p = 1$.

We may think of the Configuration model as the Erdös-Réyni random graphs $G(n,p)$ with fixed degree distribution. Properties from the Erdös-Réyni random graphs survive in the Configuration model; such as phase transition and connectedness, amongst other things. In this respect, the Configuration model looks very much like Erdös-Réyni. It is in fact the case that both Erdös-Réyni and the Configuration model both *locally* look like trees. Let us consider the following example.

7.1.1 Random d-Regular Graphs

A random graph G is said to be *d-regular* if all of its nodes have degree d.

- If $d = 1$, every node has one edge. Therefore, a random 1-regular graph G is a collection of disjoint pairs.
- If $d = 2$, G is a collection of disjoint cycles. This graph is almost surely not connected (i.e., $\mathbb{P}$ (G is connected) $= 0$).
- If $d = 3$, we have that G is a connected graph with probability not equal to zero. That is, $\mathbb{P}$ (G is connected) $\neq 0$.

For $d \geq 3$, we have the following theorem:

Theorem 7.1. *Let $d \in \mathbb{N}$ such that $d \geq 3$. If G is a random d-regular graph, then G is γ - expander.*

7.2 γ - Expander Graphs

Let V be a set of vertices, E be a set of edges and $\gamma > 0$. A graph $G(V,E)$ is said to be *γ-expander* if for any $A \subset V$ with $|A| < \frac{1}{2}|V|$, we have

$$|\partial(A, A^c)| \geq \gamma |A| \tag{7.1}$$

where $\partial(A, A^c)$ denotes the set of edges between A and A^c, and $|A|$ denotes the size of A. With this definition, we may sketch a idea of a proof for the following lemma.

Lemma 7.2. *Suppose $G(V, E)$ is a γ-expander graph with degree at most d. Then,*

$$D(G) \leq \frac{2d}{\gamma} \log(n) + 1 \tag{7.2}$$

where $D(G)$ denotes the diameter of the graph G.

Proof: (*Idea/Sketch*) Let $S_0 = \{s\}$ be a root node and define

$$S_{j+1} = S_j \cup N(S_j) \tag{7.3}$$

where $N(S_j) = \{v \in V : v \notin S_j \text{ and there exists } u \in S_j \text{ such that } (u, v) \in E\}$. That is, $N(S_j)$ is the set of all neighbors of elements in S_j. So then,

$$|S_j| = |S_j| + |N(S_j)| \tag{7.4}$$
$$\geq |S_j| + \gamma|S_j| \tag{7.5}$$
$$= (1 + \gamma)|S_j| \tag{7.6}$$

for $|S_j| \leq \frac{1}{2}|V|$. Hence,

$$|S_{j+1}| \geq (1 + \gamma)|S_j| \geq \cdots \geq (1 + \gamma)^{j+1}|S_0| = (1 + \gamma)^{j+1} \tag{7.7}$$

This implies that $D(G) = O(\log(n))$. □

Example: Let $\mathcal{T}$ be the tree for which each node at generation n has only two offspring stemming from it, where $n \in \mathbb{N}$; and consider the lattice $\mathcal{L}$. It follows that $\mathcal{T}$ is γ-expander with $\gamma = 1$. However, $\mathcal{L}$ is *not* γ-expander. To see this, consider a circle of radius R centered at the point p. It turns out that $|\partial(C(R))| \approx 2\pi R$ and $|C(R)| \approx \pi R^2$ for large R. Now consider the quotient

$$Q = \frac{|\partial(C(R))|}{|C(R)|}. \tag{7.8}$$

It is easy to see that $Q \to 0$ as $R \to \infty$. Therefore, $\mathcal{L}$ is not γ-expander. From this we see that expander graphs (locally) look more like trees, rather than lattices.

7.3 Strong and Weak Ties

Suppose we wanted to ask the question

How do people find and switch to new jobs?

Is it through close friends (strong ties) or through acquaintances (weak ties)? It happens to be the case that weak ties is the correct answer to this question. A detailed discussion of the strength of weak ties can be found in the Granovetter article on the course webpage.

7.3.1 Strong Triadic Closure

The Strong Triadic Closure Property states that if a node has strong ties to two neighbors, then these neighbors must have at least a weak tie between them. We could also require a probabilistic version of this. Let E be a set of edges and E' be the set of edges with strong ties. The the Strong Triadic Closure Property is satisfied when

$$\mathbb{P}((j,k) \in E | (i,j) \in E', (i,k) \in E') > \mathbb{P}((j,k) \in E). \tag{7.9}$$

Said in another way, if a node has strong ties to two neighbors, then these neighbors must have at least a weak tie between them. Notice that from the probabilistic version we can show that we are more likely to find a job through acquaintances as opposed to a close friend.

Now suppose that $e \in E$ is an edge between nodes A and B. If A and B have no common acquaintances or friends, then e is said to be a *local bridge*. If e is a local bridge then we define the *span* of e as being the length of the path $d(A, B)$ if e is removed. Notice that this implies that the span of a local bridge must be at least equal to three.

Theorem 7.3. *Let A be a node and suppose the strong triadic closure property is satisfied. If A has at least two strong ties, then any local bridge of A is a weak tie.*

Proof: Take some network, and consider a node A that satisfies the strong triadic closure property and is involved in at least two strong ties. Now suppose A is involved in a local bridge say, to a node B that is a strong tie. Since A is involved in at least two strong ties, and the edge to B is only one of them, it must have a strong tie to some other node, which we'll call C. Since the (A, B) edge is a local bridge, A and B must have no friends in common, and so the (B, C) edge must not exist. But this contradicts strong triadic closure, which says that since the (A, B) and (A, C) edges are both strong ties, the (B, C) edge must exist. This contradiction shows that our initial premise, the existence of a local bridge that is a strong tie, cannot hold, finishing the argument. □

7.4 Clustering

Suppose we wanted to keep track of the proportion of strong ties in a network. The way we do this is by using the *clustering ceofficient* C. It is defined by

$$C = \frac{3T}{\tau} \tag{7.10}$$

where T is the number of triangles and τ is the number of triads. The clustering coefficient for a single node u is given by

$$C_u = \frac{2|\{v, w \in N(u) : (v, w) \in E\}|}{|N(u)|(|N(u)| - 1)} \tag{7.11}$$

where $N(u)$ is the set of neighbors of u and E is the set of edges in the network.

Remark: The clustering coefficient of the Erdös-Réyni random graph model $G(n,p)$ goes to zero as $n \to \infty$. This is because

$$C = \frac{3\mathbb{E}[T]}{\mathbb{E}[\tau]} \propto \frac{p(n)^3}{p(n)^2} = p(n) \to 0 \tag{7.12}$$

as $n \to \infty$, where $p(n) = \frac{\lambda}{n}$ with λ a constant.

Now, to capture strong ties in a model we need to move from these unstructured models to more structured models.

7.5 Watts - Strogratz small-world model

The idea is to start with a lattice-like structure. For example, let us take the ring lattice $\mathcal{R}$. This structure, as it is, has high clustering and a large diameter which is on the order of $\Theta(n)$. Now what we do is, at random, add in short-cuts to other nodes. Doing this also gives us high clustering but, a small diameter on the order of $O(\log(n))$. The result is what is known as the *Watts - Strogratz* small-world model.

Let p denote the associated rewiring probability. If $p = \epsilon > 0$ for ϵ small, we get this model. If $p = 1$, we get a random d-regular graph which implies low clustering and small diameter.

Theorem 7.4. *The Watts - Strogratz small-world model satisfies the condition: There exists a constant C such that*

$$\mathbb{P}(D > C\log(n)) \to 0 \tag{7.13}$$

as $n \to \infty$, where D denotes the diameter.

References

1. Easley, D., and Kleinberg, J. (2010). Networks, Crowds, and Markets: Reasoning about a Highly Connected World. Cambridge University Press, 47-56.

2. Granovetter, M. S. (1973). The strength of weak ties. American Journal of Sociology, 1360-1380.

3. Watts, D. J., and Strogratz, S. H. (1998). Collective dynamics of 'small-world' networks. Nature, 393(6684), 440-442.