M375T/M396C: Topics in Complex Networks | Spring 2013

Lecture 9 — February 12

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9.1 Watts-Strogatz small-world model

Definition A routing scheme from a source $u \in V$ to a target $v \in V$ is a path {u(0),...,u(k)}, where $u(0) = u$, $u(k) = v$, and $u(j) \in N(u(j-1)) \forall j \leq k$.

Definition We say a routing is decentralized if $u(k)$ is determined only with knowledge of $\{u(0), ..., u(k)\} \cup \{N(u(0)), ..., N(u(k-1))\} \cup \{v\}$.

Our underlying model is a lattice (strong ties) with randomly added weak ties. We define the distance between two vertices of the lattice $u = (u_1, v_1)$ and $v = (v_1, v_2)$ to be

$$d(u, v) = |u_1 - v_1| + |u_2 - v_2|$$

Definition A greedy routing scheme with target v is one in which the kth node $u(k)$, is chosen such that

$$d(u(k), v) = min\{d(u, v), u \in N(u(k-1))\}$$

In other words, at each node, we choose the neighbor that gets us closest to the target in terms of taxi-cab distance.

Watts-Strogatz small-world model Start with a lattice with $n = m^2$ nodes. Each node u is connected to its neighbors:

$$\{w \in V - \{u\} : ||u - w|| = 1\}$$

Where || || is L_1(taxi-cab) distance. These represent strong ties (close friends) in a network. Each node chooses a long-range "shortcut" uniformly at random. In other words, each node will have a weak tie to another node in that lattice, and that will be chosen at random. So if $u \rightsquigarrow v$ denotes a shortcut between nodes u and v, we have

$$\mathbb{P}(u \rightsquigarrow v) = \frac{1}{|V - \{u\}|} = \frac{1}{n-1}$$

Now let $T_{alg}(u, v)$ be the time it takes to route from u to v given some algorithm alg. We want to know if $W - S$ is algorithmically small world. In other words, is $\mathbb{E}[T_{alg}(u, v)] = O(log(n))$?

Theorem 9.1. *For "most" $u, v \in V$ and any decentralized algorithm alg*

$$\mathbb{E}[T_{alg}(u,v)] = \Omega(m^{\frac{2}{3}})$$

So then we have that W-S is not algorithmically small-world. We thus want to modify it so that we have $O(log(n))$.

9.2 Kleinberg small-world model

Consider an infinite family parametrized by $\alpha \geq 0$. We will have nodes choose a long-range "shortcut" with probability:

$$\mathbb{P}(u \rightsquigarrow v) = \frac{\frac{1}{||u-v||^{\alpha}}}{\sum_{w \neq v} \frac{1}{||u-w||^{\alpha}}}$$

We call this the "Kleinberg Model"

Note that α is a clustering parameter for weak ties. Also, if $\alpha = 0$, we get the W-S small world model, while if $\alpha = \infty$, we get $\mathbb{P}(u \rightsquigarrow v) = \frac{1}{N(u)}$, where $N(u)$ denotes the set of neighbors of u.

9.2.1 Choosing α

We want a model that is algorithmically small-world. How can we choose α so that the Kleinberg Model is algorithmically small-world?

Idea When α is too small, weak ties spread too thin, but when α gets large, the shortcuts don't make a difference.

We want to find $\alpha \in (0, \infty)$ such that we get the the algorithmic small world property.

Theorem 9.2. *If $\alpha = 2$, then with the greedy algorithm,*

$$\mathbb{E}[T_{greedy}(u,v)] = O(log(n))$$

Proof: idea: Pick some target v. We take a box around it of size $2s$, so there are $2s$ nodes in the box. Then if $u \notin box$, we find how long it takes on average to get to $\frac{||u-v||}{2^j}$, $j = 1, 2, ..., log_2(m)$. □

Theorem 9.3.

i) If $\alpha \in [0, 2)$, then for "most" pairs (u, v), (proportion of pairs for which this holds goes to 1 as $n \to \infty$), and any decentralized algorithm alg,

$$\mathbb{E}[T_{alg}(u,v)] = \Omega(m^{\frac{(2-\alpha)}{3}})$$

ii) If $\alpha > 2$,

$$\mathbb{E}[T_{alg}(u,v)] = \Omega(m^{\frac{(\alpha-2)}{\alpha-1}})$$

Proof: Consider

$$z = \sum_{w \neq u} \frac{1}{||u-w||^{\alpha}} = \sum_{i=1}^{2m} \sum_{w:||u-w||=i} \frac{1}{i^{\alpha}} \sim \sum_{i=1}^{2m} i \cdot i^{\alpha}$$

Now

$$\int_1^m x \cdot x^{-\alpha} dx = \int_1^m x^{1-\alpha} dx \sim m^{2-\alpha} \ when \ \alpha \neq 2 \ and \ log(m) \ when \ \alpha = 2$$

Idea for (i): Choose some destination v and consider a box of side length m^{β} for some given $\beta < 1$. Take an m^{β} box around v. Then we have that the proportion of sources $u \in V$ such that u is inside the box is $\frac{m^{2\beta}}{m^2} = m^{2(\beta-1)} \Rightarrow$ as $n \to \infty$, the proportion of such sources $u \to 0$.

Now consider $u \in V$ outside of the box. Without any shortcuts, we'll need to take at least $\Omega(m^{\beta})$ steps to get into the box. Then if $\mathbb{P}$ is the probability that u has a shortcut into the box,

$$\mathbb{P} \leq \frac{sum_{w \in box} \frac{1}{||u-v||^{\alpha}}}{z} \leq \frac{number \ of \ nodes \ in \ box}{z} = \frac{m^{2\beta}}{z} \sim \frac{m^{2\beta}}{m^{2-\alpha}} = m^{2\beta-2+\alpha}$$

$\Rightarrow$ On average, we will have to check $m^{-2\beta+2-\alpha}$ nodes to find a shortcut into the box. Then the minimum amount of time to route from u to $v \geq \Omega\{max\{m^{\beta}, m^{-2\beta+2-\alpha}\}\}$. We want to minimize this over β, which we can do by finding β such that $\beta = -2\beta + 2 - \alpha$:

$$\beta = -2\beta + 2 - \alpha \Rightarrow 3\beta = 2 - \alpha \Rightarrow \beta = \frac{2-\alpha}{3}$$

Then we have that $\mathbb{E}[T_{alg}(u,v)] = \Omega(m^{\frac{2-\alpha}{3}})$. □