M341 (92150), Sample Final Exam Solutions

1. Consider $S = \{[2,-3,4,-1]^T, [-6,9,-12,3]^T, [3,1,-2,2]^T, [2,8,-12,3]^T, [7,6,-10,4]^T\}$.

a) Is S linearly independent? If not, find a maximal linearly independent subset.

Solution: Let $A = \begin{bmatrix} 2 & -6 & 3 & 2 & 7 \\ -3 & 9 & 1 & 8 & 6 \\ 4 & -12 & -2 & -12 & -10 \\ -1 & 3 & 2 & 3 & 4 \end{bmatrix}$ be the matrix whose columns are vectors in S. Then $\operatorname{rref}(A) = \begin{bmatrix} 1 & -3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, which does not have a pivot in each column so S is not linearly independent. One maximal linearly independent subset consists of the pivot columns of A—i.e., $B = \{[2,-3,4,-1]^T, [3,1,-2,2]^T, [2,8,-12,3]^T\}$.

b) Does S span $\mathbb{R}^4$? If not, express span(S) in terms of a minimal spanning set.

Solution: No, S does not span $\mathbb{R}^4$ since rref(A) does not have a pivot in every row. A minimal spanning subset of S is the set B found in part (a), and $\operatorname{span}(S) = \operatorname{span}(B)$.

c) Construct a basis for span(S). What is dim (span(S))?

Solution: B forms a basis for span(S), and $\dim(\operatorname{span}(S)) = |B| = 3$.

d) Construct a basis for $\mathbb{R}^4$ that contains the maximal linearly independent subset found in part (a).

Solution: We must extend the linearly independent set B by adding to it another vector that is linearly independent to B. For example, let $\boldsymbol{v} = [1, 0, 0, 0]^T$ and define $\tilde{B} = B \cup \{\boldsymbol{v}\}$. Putting the vectors in $\tilde{B}$ as columns of a matrix $\tilde{A}$ we find that $\operatorname{rref}\left(\tilde{A}\right) = I_4$ so $\tilde{B}$ is a basis of $\mathbb{R}^4$.

2. Prove that all vectors orthogonal to $[2, -3, 1]^T$ forms a subspace $\mathcal{W}$ of $\mathbb{R}^3$. What is dim ($\mathcal{W}$) and why?

Solution: Let $\boldsymbol{v} = [2, -3, 1]^T$. Note that $\mathbf{0} \in \mathcal{W}$ since $\mathbf{0} \cdot \boldsymbol{v} = 0$ so $\mathcal{W}$ is nonempty. Now suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{W}$ and c is a scalar. Then $(\boldsymbol{x}+\boldsymbol{y})\cdot\boldsymbol{v} = (\boldsymbol{x}\cdot\boldsymbol{v}) + (\boldsymbol{y}\cdot\boldsymbol{v}) = 0+0 = 0$ and $(c\boldsymbol{x})\cdot\boldsymbol{v} = c(\boldsymbol{x}\cdot\boldsymbol{v}) = c0 = 0$.

We will compute $\mathcal{W}$ explicitly in order to find its dimension. Since $\boldsymbol{x} = [x_1, x_2, x_3]^T \in \mathcal{W}$ if and only if $[2, -3, 1]^T \cdot \boldsymbol{x} = 2x_1 - 3x_2 + x_3 = 0$, we have that $x_3 = -2x_1 + 3x_2$ so $\boldsymbol{x} = x_1[1, 0, -2]^T + x_2[0, 1, 3]^T$. Therefore, $B = \{[1, 0, -2]^T, [0, 1, 3]^T\}$ is a basis for $\mathcal{W}$ and $\dim(\mathcal{W}) = 2$.

3. Let $\mathcal{V}$ be an n-dimensional vector space and $\mathcal{W}$ be an m-dimensional vector space.

a) Suppose $n < m$. Show that there is no linear transformation $L\colon \mathcal{V} \to \mathcal{W}$ such that L is onto.

Solution: Similar to part (a), suppose $L\colon \mathcal{V} \to \mathcal{W}$ is linear. Since $\dim(\operatorname{Ker}(L)) \geq 0$,

$$\dim(\operatorname{Range}(L)) = \dim(\mathcal{V}) - \dim(\operatorname{Ker}(L)) = n - \dim(\operatorname{Ker}(L)) \leq n < m = \dim(\mathcal{W}).$$

So $\text{Range}(L) \neq \mathcal{W}$ and L is not onto.

b) Suppose $n > m$. Show that there is no linear transformation $L: \mathcal{V} \to \mathcal{W}$ such that L is one-to-one.

Solution: Suppose $L: \mathcal{V} \to \mathcal{W}$ is linear. Since $\dim(\text{Range}(L)) \leq \dim(\mathcal{W}) = m$, the dimension theorem implies that

$$\dim(\text{Ker}(L)) = \dim(\mathcal{V}) - \dim(\text{Range}(L)) = n - \dim(\text{Range}(L)) \geq n - m > 0.$$

Therefore, $\dim(\text{Ker}(L)) \neq 0$ so L cannot be one-to-one.

c) Prove that $L: \mathcal{V} \to \mathcal{W}$ is an isomorphism only if $n = m$.

Solution: We must have that $n \geq m$ in order for L to be onto, and $n \leq m$ in order for L to be one-to-one. Therefore, for L to be both one-to-one and onto (i.e., an isomorphism) we must have that $n = m$.

d) Let $\mathcal{U}_3$ be the space of 3×3 upper triangular matrices and define the linear transformation $L: \mathcal{U}_3 \to \mathcal{M}_{33}$ by $L(A) = \frac{1}{2}(A + A^T)$. Is L onto? Is L one-to-one?

Solution: Since $\dim(\mathcal{U}_3) = 6 < 9 = \dim(\mathcal{M}_{33})$, L cannot be onto by part (a). Part (b) does not help us determine whether L is one-to-one, so we must determine this directly. In order for $A \in \text{Ker}(L)$, we must have that $\frac{1}{2}(A + A^T) = 0$—i.e., $A = -A^T$. Since A is upper-triangular, this implies that $A = O_{33}$ (why?). Therefore, $\text{Ker}(L) = \{O_{33}\}$ so dim $(\text{Ker}(L)) = 0$ and L is one-to-one.

4. Let $\mathcal{V} = \mathcal{P}_2$ with standard basis $B = \{1, x, x^2\}$ and let $\mathcal{W} = \mathcal{M}_{22}$ with standard basis $D = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Consider $L: \mathcal{V} \to \mathcal{W}$ given by

$$L(p) = \begin{bmatrix} p(1) - p(0) & p(2) - p(0) \\ p(-1) - p(0) & p(-2) - p(0) \end{bmatrix}.$$

(For example, $L(x^2) = \begin{bmatrix} 1^2 - 0^2 & 2^2 - 0^2 \\ (-1)^2 - 0^2 & (-2)^2 - 0^2 \end{bmatrix}$.)

a) Prove that L is a linear transformation.

Solution: Straightforward to show $L(p + q) = L(p) + L(q)$ and $L(cp) = cL(p)$.

b) Find the matrix representation $[L]_{DB}$ of L with respect to the bases B and D.

Solution: Since $L(1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $L(x) = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, $L(x^2) = \begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}$ the matrix of L is

$$[L]_{DB} = \left([L(1)]_D \ [L(x)]_D \ [L(x^2)]_D \right) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & -2 & 4 \end{bmatrix}.$$

c) What is the dimension of $\text{Ker}(L)$? Find a basis for $\text{Ker}(L)$.

Solution: The reduced row-echelon form of $[L]_{DB}$ is

$$\operatorname{rref}\big([L]_{DB}\big) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

There is one column without a pivot, which gives one free variable. Therefore, dim $(\operatorname{Ker}(L)) = 1$. A basis for the kernel of $[L]_{DB}$ is $[1,\ 0,\ 0]^T$, so a basis for $\operatorname{Ker}(L)$ is the polynomial $p(x) = 1$.

d) What is the dimension of Range(L)? Find a basis for Range(L).

Solution: Since there two pivot columns in $\operatorname{rref}\big([L]_{DB}\big)$, $\dim(\operatorname{Range}(L)) = 2$. A basis for the range of $[L]_{DB}$ consists of the pivot columns of $[L]_{DB}$, i.e., $\{[1,\ 2,\ -1,\ -2]^T,\ [1,\ 4,\ 1,\ 4]^T\}$. This gives the basis $\left\{\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}\right\}$ for Range(L).

e) Verify the dimension theorem (i.e., rank-nullity theorem) for L.

Solution: $\dim(\operatorname{Ker}(L)) + \dim(\operatorname{Range}(L)) = 1 + 2 = 3 = \dim(\mathcal{P}_2)$.

5. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 7 \\ 5 & 10 & 13 & 18 \end{bmatrix}$ so that $\operatorname{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

a) Are the vectors $[1,\ 2,\ 5]^T$, $[2,\ 4,\ 10]^T$, $[3,\ 5,\ 13]^T$, $[4,\ 7,\ 18]^T$ linearly independent? Do they span $\mathbb{R}^3$?

Solution: The vectors given are the columns of A. Since $\operatorname{rref}(A)$ has columns without pivots, the columns of A are not linearly independent. Since $\operatorname{rref}(A)$ has a row without a pivot, the columns of A do not span $\mathbb{R}^3$.

b) Find a basis for the span of the four vectors in part (a).

Solution: A basis for the column space of A is given by the pivot columns of A, namely, $[1,2,5]^T$ and $[3,5,13]^T$.

6. Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 8x_1 - 10x_2 \\ 3x_1 - 3x_2 \end{bmatrix}$. Define the standard basis $B = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ and an alternate basis $D = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}\right\}$. Consider a vector $\boldsymbol{v} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$.

a) Find the change of basis matrices P_{DB} (i.e., from B to D) and P_{BD} (i.e., from D to B).

Solution: $P_{BD} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $P_{DB} = P_{BD}^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

b) Compute $[\boldsymbol{v}]_B$ and $[\boldsymbol{v}]_D$.

Solution: $[\boldsymbol{v}]_B = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$, $[\boldsymbol{v}]_D = P_{DB}[\boldsymbol{v}]_B = \begin{bmatrix} 9 \\ -2 \end{bmatrix}$.

c) Find $[L]_{BB}$ and $[L]_{DD}$.

Solution: $[L]_{BB} = \begin{bmatrix} 8 & -10 \\ 3 & -3 \end{bmatrix}$, $[L]_{DD} = P_{DB}[L]_{BB}P_{BD} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

7. Consider $A = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$.

a) Write the characteristic polynomial $p_A(\lambda)$ and use this to determine the eigenvalues of A. [Hint: Factor out a term of the form $(\lambda - 1)$.]

Solution: Since $p_A(\lambda) = \det(A - \lambda I) = -\lambda\big(\lambda^2 - 1\big) + 2(\lambda - 1) = (\lambda + 2)(\lambda - 1)^2$, A has eigenvalues -2 and 1.

b) Find the eigenspaces corresponding to the eigenvalues of A.

Solution: The corresponding eigenspaces are $E_{-2} = \operatorname{span}\big\{[1,\, 1,\, 1]^T\big\}$, $E_1 = \operatorname{span}\big\{[1,\, -1,\, 0]^T, [1, 0, -1]^T\big\}$.

c) Is A diagonalizable? If so, find a nonsingular matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution: $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$.

8. Let $A = \begin{bmatrix} -8 & 4 & -3 & 2 \\ 2 & 1 & -1 & 0 \\ -3 & -5 & 4 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix}$.

a) Find the determinant of A using a cofactor expansion.

Solution: Expanding about the last column, we get

$$\det(A) = (-1)^{1+4} 2 \begin{vmatrix} 2 & 1 & -1 \\ -3 & -5 & 4 \\ 2 & -4 & 3 \end{vmatrix} + (-1)^{4+4}(-1) \begin{vmatrix} -8 & 4 & -3 \\ 2 & 1 & -1 \\ -3 & -5 & 4 \end{vmatrix} = 6 - 9 = -3.$$

b) Find the determinant of A by using row operations to put A into upper triangular form. Verify that your answer agrees with part (a).

Solution: We perform row operations that put A into upper triangular form in a manner that avoids introducing fractions:

$$A = \begin{bmatrix} -8 & 4 & -3 & 2 \\ 2 & 1 & -1 & 0 \\ -3 & -5 & 4 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix} \xrightarrow[\langle 3\rangle \leftarrow -2\langle 3\rangle]{\langle 1\rangle \leftrightarrow \langle 2\rangle} \begin{bmatrix} 2 & 1 & -1 & 0 \\ -8 & 4 & -3 & 2 \\ 6 & 10 & -8 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix} \xrightarrow[\substack{\langle 2\rangle \leftarrow \langle 2\rangle + 4\langle 1\rangle \\ \langle 3\rangle \leftarrow \langle 3\rangle - 3\langle 2\rangle \\ \langle 4\rangle \leftarrow \langle 4\rangle - \langle 1\rangle}]{} \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 8 & -7 & 2 \\ 0 & 7 & -5 & 0 \\ 0 & -5 & 4 & -1 \end{bmatrix}$$

$$\xrightarrow[\substack{\langle 2\rangle \leftarrow \langle 2\rangle - \langle 3\rangle \\ \langle 4\rangle \leftarrow \langle 4\rangle + \langle 3\rangle}]{} \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 7 & -5 & 0 \\ 0 & 2 & -1 & -1 \end{bmatrix} \xrightarrow[\substack{\langle 3\rangle \leftarrow \langle 3\rangle - 7\langle 2\rangle \\ \langle 4\rangle \leftarrow \langle 4\rangle + 2\langle 2\rangle}]{} \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 9 & -14 \\ 0 & 0 & 3 & -5 \end{bmatrix} \xrightarrow[\langle 3\rangle \leftrightarrow \langle 4\rangle]{} \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 3 & -5 \\ 0 & 0 & 9 & -14 \end{bmatrix}$$

$$\xrightarrow[\langle 4\rangle \leftarrow \langle 4\rangle - 3\langle 3\rangle]{} \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 3 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U.$$

Therefore, $6 = \det(U) = (-1) \times (-2) \times (-1) \times \det(A)$ so $\det(A) = -3$ as we found before.

c) Is A nonsingular (i.e., invertible)?

Solution: Yes, A is nonsingular since $\det(A) \neq 0$.

9. The following two questions are unrelated to each other.

a) Show that $\mathcal{V} = \mathbb{R}$ with the usual operation of scalar multiplication but with addition given by $x \oplus y = 2(x + y)$ is not a vector space.

Solution: Several of the axioms fail in this case. For example, $3 \oplus (4 \oplus 5) = 3 \oplus 18 = 42$ but $(3 \oplus 4) \oplus 5 = 14 \oplus 5 = 38$.

b) Consider the subset $\mathcal{S}$ of all matrices in $\mathcal{M}_{55}$ which have eigenvalue 1. Is $\mathcal{S}$ a *subspace* of $\mathcal{M}_{55}$? Explain why or why not.

Solution: No, $\mathcal{S}$ is not a subspace since it is not closed under scalar multiplication. To see this, note that $I_5 \in \mathcal{S}$ but that $2I_5 \notin \mathcal{S}$ since $2I_5$ only has $\lambda = 2$ as an eigenvalue.

10. True or false? Explain your answers.

a) The plane $x_1 + 3x_2 - 4x_3 = 1$ is a subspace of $\mathbb{R}^3$.

Solution: False. The zero element does not lie in the plane so it cannot be a subspace.

b) If A is a 3×5 matrix, then $\dim(\mathrm{Ker}(A)) \geq 2$.

Solution: True. There are at most 3 pivots in rref(A), so there are at least two columns which give free variables.

c) Let $B = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ be a basis for a vector space $\mathcal{V}$. If n vectors $\{\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n\}$ span V then the coordinate vectors $\left\{[\boldsymbol{d}_1]_B, \ldots, [\boldsymbol{d}_n]_B\right\}$ are linearly independent.

Solution: True. Whenever the number of vectors is the same as the dimension of the space, either the vectors both span and are linearly independent, or fail to span and are linearly dependent. Since the vectors $\{\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n\}$ span $\mathcal{V}$, they are linearly independent in V. Therefore, their coordinate representations $\left\{[\boldsymbol{d}_1]_B, \ldots, [\boldsymbol{d}_n]_B\right\}$ are linearly independent in $\mathbb{R}^n$.

d) Every linear transformation $L: \mathbb{R}^5 \to \mathbb{R}^4$ takes the form $L(\boldsymbol{x}) = A\boldsymbol{x}$ with A a 5×4 matrix.

Solution: False. Every linear transformation $L: \mathbb{R}^5 \to \mathbb{R}^4$ is given by a 4×5 matrix.

e) Let rref(A) be the reduced row-echelon form of a matrix A. Then, the pivot columns of rref(A) form a basis of the column space of A (i.e., the span of the columns of A).

Solution: False. The pivot columns of the *original* matrix A form a basis of the column space of A.

f) The vectors $\boldsymbol{b}_1 = 1 + t + 2t^2$, $\boldsymbol{b}_2 = 2 + 3t + 5t^2$, $\boldsymbol{b}_3 = 3 + 7 + 9t^2$ form a basis for $\mathcal{P}_2$.

Solution: True. Using the standard basis, the coordinates of the given vectors are $[1, 1, 2]^T$, $[2, 3, 5]^T$, and $[3, 7, 9]^T$. Since the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 2 & 5 & 9 \end{bmatrix}$ row-reduces to the identity, these vectors form a basis.

g) [Harder...] The equation $p''(t) - p(t) = q(t)$ has a solution $p \in \mathcal{P}_3$ for any $q \in \mathcal{P}_3$.

Solution: True. To see this, we write the equation as $L(p(x)) = q(x)$ where $L = \frac{d^2}{dx^2} - I$ is a linear operator on $\mathcal{P}_3$. The matrix representation of L in the standard basis $B = \{1, x, x^2, x^3\}$ is

$$[L]_{BB} = \left(\ [L(1)]_B \ \ [L(x)]_B \ \ \left[L(x^2)\right]_B \ \ \left[L(x^3)\right]_B \ \right) = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Since this row-reduces to the identity, we have that $[L]_{BB}$ has full rank and, therefore, L is onto. This immediately implies that $L(p) = q$ has a solution $p \in \mathcal{P}_3$ for any given $q \in \mathcal{P}_3$. Furthermore, the coordinates $[p]_B$ of the solution are found by solving the matrix equation $[L]_{BB}[p]_B = [q]_B$.