M341 (92150), Sample Final Exam Questions

Below are some sample final exam questions. Please note that the intent of these is to help pre-pare for the exam, and that actual exam questions will not merely be modifications of these prob-lems. Consult your HW, lecture notes, and books for additional sources of material to review. Finally, to obtain the full effect of an exam please complete these problems under time-pressure.

1. Consider $S = \left\{ [2, -3, 4, -1]^T, [-6, 9, -12, 3]^T, [3, 1, -2, 2]^T, [2, 8, -12, 3]^T, [7, 6, -10, 4]^T \right\}$.

   a) Is S linearly independent? If not, find a maximal linearly independent subset.

   b) Does S span $\mathbb{R}^4$? If not, express span(S) in terms of a minimal spanning set.

   c) Construct a basis for span(S). What is $\dim(\text{span}(S))$?

   d) Construct a basis for $\mathbb{R}^4$ that contains the maximal linearly independent subset found in part (a).

2. Prove that all vectors orthogonal to $[2, -3, 1]^T$ forms a subspace $\mathcal{W}$ of $\mathbb{R}^3$. What is $\dim(\mathcal{W})$ and why?

3. Let $\mathcal{V}$ be an n-dimensional vector space and $\mathcal{W}$ be an m-dimensional vector space.

   a) Suppose $n < m$. Show that there is no linear transformation $L: \mathcal{V} \to \mathcal{W}$ such that L is onto.

   b) Suppose $n > m$. Show that there is no linear transformation $L: \mathcal{V} \to \mathcal{W}$ such that L is one-to-one.

   c) Prove that $L: \mathcal{V} \to \mathcal{W}$ is an isomorphism only if $n = m$.

   d) Let $\mathcal{U}_3$ be the space of 3×3 upper triangular matrices and define the linear transformation $L: \mathcal{U}_3 \to \mathcal{M}_{33}$ by $L(A) = \frac{1}{2}(A + A^T)$. Is L onto? Is L one-to-one?

4. Let $\mathcal{V} = \mathcal{P}_2$ with standard basis $B = \{1, x, x^2\}$ and let $\mathcal{W} = \mathcal{M}_{22}$ with standard basis $D = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Consider $L: \mathcal{V} \to \mathcal{W}$ given by

$$L(p) = \begin{bmatrix} p(1) - p(0) & p(2) - p(0) \\ p(-1) - p(0) & p(-2) - p(0) \end{bmatrix}.$$

(For example, $L(x^2) = \begin{bmatrix} 1^2 - 0^2 & 2^2 - 0^2 \\ (-1)^2 - 0^2 & (-2)^2 - 0^2 \end{bmatrix}$.)

   a) Prove that L is a linear transformation.

   b) Find the matrix representation $[L]_{DB}$ of L with respect to the bases B and D.

   c) What is the dimension of Ker(L)? Find a basis for Ker(L).

   d) What is the dimension of Range(L)? Find a basis for Range(L).

e) Verify the dimension theorem (i.e., rank-nullity theorem) for L.

5. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 7 \\ 5 & 10 & 13 & 18 \end{bmatrix}$ so that $\mathrm{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

a) Are the vectors $[1, 2, 5]^T$, $[2, 4, 10]^T$, $[3, 5, 13]^T$, $[4, 7, 18]^T$ linearly independent? Do they span $\mathbb{R}^3$?

b) Find a basis for the span of the four vectors in part (a).

6. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 8x_1 - 10x_2 \\ 3x_1 - 3x_2 \end{bmatrix}$. Define the standard basis $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and an alternate basis $D = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}$. Consider a vector $v = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$.

a) Find the change of basis matrices P_{DB} (i.e., from B to D) and P_{BD} (i.e., from D to B).

b) Compute $[v]_B$ and $[v]_D$.

c) Find $[L]_{BB}$ and $[L]_{DD}$.

7. Consider $A = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$.

a) Write the characteristic polynomial $p_A(\lambda)$ and use this to determine the eigenvalues of A. [Hint: Factor out a term of the form $(\lambda - 1)$.]

b) Find the eigenspaces corresponding to the eigenvalues of A.

c) Is A diagonalizable? If so, find a nonsingular matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

8. Let $A = \begin{bmatrix} -8 & 4 & -3 & 2 \\ 2 & 1 & -1 & 0 \\ -3 & -5 & 4 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix}$.

a) Find the determinant of A using a cofactor expansion.

b) Find the determinant of A by using row operations to put A into upper triangular form. Verify that your answer agrees with part (a).

c) Is A nonsingular (i.e., invertible)?

9. The following two questions are unrelated to each other.

a) Show that $\mathcal{V} = \mathbb{R}$ with the usual operation of scalar multiplication but with addition given by $x \oplus y = 2(x + y)$ is not a vector space.

b) Consider the subset S of all matrices in $\mathcal{M}_{55}$ which have eigenvalue 1. Is S a *subspace* of $\mathcal{M}_{55}$? Explain why or why not.

10. True or false? Explain your answers.

   a) The plane $x_1 + 3x_2 - 4x_3 = 1$ is a subspace of $\mathbb{R}^3$.

   b) If A is a 3×5 matrix, then $\dim(\text{Ker}(A)) \geq 2$.

   c) Let $B = \{\boldsymbol{b}_1, ..., \boldsymbol{b}_n\}$ be a basis for a vector space $\mathcal{V}$. If n vectors $\{\boldsymbol{d}_1, ..., \boldsymbol{d}_n\}$ span V then the coordinate vectors $\{[\boldsymbol{d}_1]_B, ..., [\boldsymbol{d}_n]_B\}$ are linearly independent.

   d) Every linear transformation $L : \mathbb{R}^5 \to \mathbb{R}^4$ takes the form $L(\boldsymbol{x}) = A\boldsymbol{x}$ with A a 5×4 matrix.

   e) Let rref(A) be the reduced row-echelon form of a matrix A. Then, the pivot columns of rref(A) form a basis of the column space of A (i.e., the span of the columns of A).

   f) The vectors $\boldsymbol{b}_1 = 1 + t + 2t^2$, $\boldsymbol{b}_2 = 2 + 3t + 5t^2$, $\boldsymbol{b}_3 = 3 + 7 + 9t^2$ form a basis for $\mathcal{P}_2$.

   g) [Harder...] The equation $p''(t) - p(t) = q(t)$ has a solution $p \in \mathcal{P}_3$ for any $q \in \mathcal{P}_3$.