M341 (92150), Sample Midterm #1 Solutions

1. Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are said to be *parallel* if one is a scalar multiple of the other. Now consider the statement

$$\text{“If } \|\boldsymbol{x}+\boldsymbol{y}\| \neq \|\boldsymbol{x}\|+\|\boldsymbol{y}\|, \text{ then } \boldsymbol{x} \text{ is not parallel to } \boldsymbol{y}.\text{”}$$

a) Is this statement true or false? Justify your answer with a proof (if true) or a counterexample (if false).

Solution: False. Suppose $\boldsymbol{x} = [1, 0]^T$ and $\boldsymbol{y} = [-1, 0]^T$. Then $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{0}$ so $\|\boldsymbol{x} + \boldsymbol{y}\| \neq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$, but $\boldsymbol{x}$ and $\boldsymbol{y}$ are parallel to one another since one is a scalar multiple of the other.

b) State the contrapositive of the statement. Is the contrapositive true or false, and why?

Solution: The contrapositive is

$$\text{“If } \boldsymbol{x} \text{ is parallel to } \boldsymbol{y}, \text{ then } \|\boldsymbol{x}+\boldsymbol{y}\| = \|\boldsymbol{x}\|+\|\boldsymbol{y}\|.\text{”}$$

The contrapositive is false since it is equivalent to the original statement, which is false.

c) State the converse and inverse of the statement. Are the converse and inverse true or false? Justify your answer with a proof or counterexample. [Hint: You may use the fact that if $\boldsymbol{x}\cdot\boldsymbol{y} = \|\boldsymbol{x}\|\|\boldsymbol{y}\|$ then $\boldsymbol{y}$ is a *positive* scalar multiple of $\boldsymbol{x}$, as we proved in lecture.]

Solution: The converse is

$$\text{“If } \boldsymbol{x} \text{ is not parallel to } \boldsymbol{y}, \text{ then } \|\boldsymbol{x}+\boldsymbol{y}\| \neq \|\boldsymbol{x}\|+\|\boldsymbol{y}\|\text{”}$$

while the inverse is

$$\text{“If } \|\boldsymbol{x}+\boldsymbol{y}\| = \|\boldsymbol{x}\|+\|\boldsymbol{y}\|, \text{ then } \boldsymbol{x} \text{ is parallel to } \boldsymbol{y}.\text{”}$$

Either both of these statements are true or both are false, since they are equivalent. We claim that these are both true and justify it with the following proof of the inverse. Assuming $\|\boldsymbol{x}+\boldsymbol{y}\| = \|\boldsymbol{x}\|+\|\boldsymbol{y}\|$, we have that

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 = (\|\boldsymbol{x}\|+\|\boldsymbol{y}\|)^2 = \|\boldsymbol{x}\|^2 + 2\|\boldsymbol{x}\|\|\boldsymbol{y}\| + \|\boldsymbol{y}\|^2.$$

Since $\|\boldsymbol{x}+\boldsymbol{y}\|^2 = (\boldsymbol{x}+\boldsymbol{y})\cdot(\boldsymbol{x}+\boldsymbol{y}) = \|\boldsymbol{x}\|^2 + 2(\boldsymbol{x}\cdot\boldsymbol{y}) + \|\boldsymbol{y}\|^2$ we have that $\boldsymbol{x}\cdot\boldsymbol{y} = \|\boldsymbol{x}\|\|\boldsymbol{y}\|$. Therefore, by the theorem proven in lecture and mentioned in the hint above, $\boldsymbol{y}$ is a positive scalar multiple of $\boldsymbol{x}$. Therefore, $\boldsymbol{x}$ is parallel to $\boldsymbol{y}$, which completes the proof.

2. Consider the matrix $A = \begin{bmatrix} 5 & -5 & 15 \\ 4 & -2 & -6 \end{bmatrix}$.

a) Find the complete solution set of $A\boldsymbol{x} = \boldsymbol{b}$ when $\boldsymbol{b} = [40, 19]^T$.

Solution: Putting the augmented matrix $[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 5 & -5 & 15 & 40 \\ 4 & -2 & -6 & 19 \end{array}\right]$ into reduced row echelon form we obtain $\text{rref}([A|\boldsymbol{b}]) = \left[\begin{array}{ccc|c} 1 & 0 & 6 & 3/2 \\ 0 & 1 & 3 & -13/2 \end{array}\right]$. Therefore, the solution set is given by $\boldsymbol{x} = [x_1, x_2, x_3]^T$ which satisfy $x_1 = \frac{3}{2} - 6x_3$, $x_2 = -\frac{13}{2} - 3x_3$, and x_3 free.

b) What is the reduced row echelon form of A?

Solution: By the calculation above, $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 3 \end{bmatrix}$. Note that we did not need to do any extra work to obtain this.

c) What is the rank of A?

Solution: Since $\operatorname{rref}(A)$ has two pivot columns, the rank of A is 2.

d) Without performing any extra computations, does the homogeneous system $A\boldsymbol{x} = \mathbf{0}$ have nontrivial solutions? Why or why not?

Solution: Since $\operatorname{rank}(A) = 2$ is less than the number of variables $n = 3$ in the system, the homogeneous system has nontrivial solutions.

3. Let A be an $n \times n$ lower triangular matrix, and B be an $n \times n$ upper triangular matrix. Suppose A and B have no zero entries on the main diagonal, and that AB is a diagonal matrix. Prove that A must be a diagonal matrix. [Hint: Use induction on j, where $1 \leq j \leq n$ represents the j^{th} column of A. Alternatively, try a proof by contradiction.]

Solution: We give a proof by contradiction. Suppose A is lower triangular, B is upper triangular, A and B have no zeros on the diagonal, AB is diagonal, and that A is *not* a diagonal matrix. Then, there must exist an $i > j$ such that $A_{ij} \neq 0$. This implies that

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} = A_{ij} B_{jj} \neq 0$$

since $B_{kj} = 0$ when $k \neq j$ (B is diagonal) and $B_{jj} \neq 0$ (B has no zeros on the diagonal), which is a contradiction since we assumed AB is diagonal.

4. Use the following steps to find the coefficients of the circle $x^2 + y^2 + ax + by = c$ that goes through the points $(5, -1), (6, -2), (1, -7)$.

a) Write a system of linear equations that must be satisfied by the coefficients.

Solution:

$$\begin{aligned} 5a - b - c &= -26 \\ 6a - 2b - c &= -40 \\ a - 7b - c &= -50 \end{aligned}$$

b) Solve this system of equations to find the coefficients.

Solution: Reducing the corresponding augmented matrix yields

$$\left[\begin{array}{ccc|c} 5 & -1 & -1 & -26 \\ 6 & 2 & -1 & -40 \\ 1 & -7 & -1 & -50 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -7 & -1 & -50 \\ 6 & -2 & -1 & -40 \\ 5 & -1 & -1 & -26 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -7 & -1 & -50 \\ 0 & 40 & 5 & 260 \\ 0 & 34 & 4 & 224 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|c} 1 & -7 & -1 & -50 \\ 0 & 1 & 1/8 & 13/2 \\ 0 & 34 & 4 & 224 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -7 & -1 & -50 \\ 0 & 1 & 1/8 & 13/2 \\ 0 & 0 & -1/4 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -7 & -1 & -50 \\ 0 & 1 & 1/8 & 13/2 \\ 0 & 0 & 1 & -12 \end{array}\right].$$

Therefore, $c = -12$, $b = 13/2 - c/8 = 8$, and $a = -50 + 7b + c = -6$. The circle is therefore $x^2 + y^2 - 6x + 8y = -12$.

c) [Harder...] Given three distinct points in the plane, is it possible to have more than one circle go through all of them? Why or why not?

Solution: No. To see this, note that the corresponding augmented matrix is of the form

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} x_1 & y_1 & -1 & -(x_1^2+y_1^2) \\ x_2 & y_2 & -1 & -(x_2^2+y_2^2) \\ x_3 & y_3 & -1 & -(x_3^2+y_3^2) \end{array}\right]$$

where (x_1, y_1), (x_2, y_2), (x_3, y_3) are three distinct points in the plane. To have more than one solution to the system we must necessarily have that (i) the system is consistent and (ii) rref(A) has a free variable—that is, rref(A) has a column (and therefore a row) which has no pivot. However, this is not possible since

$$\left[\begin{array}{ccc|c} x_1 & y_1 & -1 & -(x_1^2+y_1^2) \\ x_2 & y_2 & -1 & -(x_2^2+y_2^2) \\ x_3 & y_3 & -1 & -(x_3^2+y_3^2) \end{array}\right] \to \left[\begin{array}{ccc|c} x_1-x_2 & y_1-y_2 & 0 & -(x_1^2-x_2^2)-(y_1^2-y_2^2) \\ x_2 & y_2 & -1 & -(x_2^2+y_2^2) \\ x_3 & y_3 & -1 & -(x_3^2+y_3^2) \end{array}\right]$$
$$\to \left[\begin{array}{ccc|c} x_1-x_2 & y_1-y_2 & 0 & -(x_1^2-x_2^2)-(y_1^2-y_2^2) \\ x_2-x_3 & y_2-y_3 & 0 & -(x_2^2-x_3^2)-(y_2^2-y_3^2) \\ x_3 & y_3 & -1 & -(x_3^2+y_3^2) \end{array}\right]$$

implies that we must either have a pivot in every row of rref(A) or that the system is inconsistent (why?).

5. True or false? Justify your answers with a short proof or counterexample.

a) If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then $A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$.

Solution: True. This can be seen by a quick computation for small values of n, and is justified by a proof by induction. The base step ($n=1$) is obviously true. For the inductive step, assume that $A^{n-1} = \begin{bmatrix} 1 & 0 \\ n-1 & 1 \end{bmatrix}$. Then

$$A^n = AA^{n-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ n-1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix},$$

which completes the proof.

b) If A and B are $n \times n$ matrices such that $AB + BA = 0$, then $A^2B^3 = B^3A^2$.

Solution: True. Since $AB = -BA$,

$$\begin{aligned} A^2B^3 &= AABBB = A(AB)BB = -ABABB = -(AB)(AB)B = -BABAB \\ &= -B(AB)(AB) = -BBABA = -BB(AB)A = BBBAA = B^3A^2. \end{aligned}$$

c) The product of two skew-symmetric matrices is also skew-symmetric.

Solution: False. Suppose A and B are skew-symmetric, so that $A^T = -A$ and $B^T = -B$. Then $(AB)^T = B^TA^T = (-B)(-A) = BA$, so AB is only skew-symmetric if and only if $AB = BA$ (that is, if A and B commute). Since two skew-symmetric matrices do not necessarily commute, the statement is false.

d) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors, then the projection of $\boldsymbol{x}$ onto $\boldsymbol{y}$ is orthogonal to the projection of $\boldsymbol{y}$ onto $\boldsymbol{x}$.

Solution: False. Take any two nonzero vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ which are not orthogonal to each other. Then $\text{proj}_{\boldsymbol{y}}\boldsymbol{x}$ is parallel to $\boldsymbol{y}$ while $\text{proj}_{\boldsymbol{x}}\boldsymbol{y}$ is parallel to $\boldsymbol{x}$, so $\text{proj}_{\boldsymbol{y}}\boldsymbol{x}$ and $\text{proj}_{\boldsymbol{x}}\boldsymbol{y}$ are not orthogonal.

e) If the homogenous system $A\boldsymbol{x}=\boldsymbol{0}$ has nontrivial (i.e., nonzero) solutions, then the nonhomogenous system $A\boldsymbol{x}=\boldsymbol{b}$ may still have a unique solution for some choice of $\boldsymbol{b}$.

Solution: False. To justify this, we need to rule out the possibility that there exists a choice of $\boldsymbol{b}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ may still have a unique solution. Note that this requires a proof and not simply a counterexample. To prove this we use a contradiction argument. Let $\boldsymbol{y} \neq \boldsymbol{0}$ be one of the nontrivial solutions of $A\boldsymbol{x} = \boldsymbol{0}$ and suppose there is some $\boldsymbol{b}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ has a *unique* solution, which we call $\boldsymbol{u}$. Then it is easy to see that $\boldsymbol{u} + \boldsymbol{y}$ is also a solution to $A\boldsymbol{x} = \boldsymbol{b}$ since $A(\boldsymbol{u} + \boldsymbol{y}) = A\boldsymbol{u} + A\boldsymbol{y} = \boldsymbol{b} + \boldsymbol{0} = \boldsymbol{b}$. But $\boldsymbol{u} \neq \boldsymbol{u} + \boldsymbol{y}$, so the solution to $A\boldsymbol{x}=\boldsymbol{b}$ cannot possibly be unique, which is a contradiction!