M341 (92150), Sample Midterm #2 Solutions

1. Let $B = \begin{bmatrix} -1 & 3 & -3 \\ 0 & -6 & 5 \\ -5 & -3 & 1 \end{bmatrix}$.

a) Show that B is invertible and compute B^{-1}.

Solution: B is invertible since when we compute the reduced row echelon form of the augmented matrix $[B|I_3]$ we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & 1 & -1/2 \\ 0 & 1 & 0 & -25/6 & -8/3 & 5/6 \\ 0 & 0 & 1 & -5 & -3 & 1 \end{array}\right]$$

which is of the form $[I_3|\cdot]$. The inverse is simply the right-hand side of this matrix:

$$B^{-1} = \begin{bmatrix} 3/2 & 1 & -1/2 \\ -25/6 & -8/3 & 5/6 \\ -5 & -3 & 1 \end{bmatrix}.$$

b) Suppose we replaced the second row $[0, -6, 5]$ of B with $[-2, 6, -6]$. Will the resulting matrix still be invertible? [Hint: There is a very quick way of finding the answer that does not require any long computations!]

Solution: The resulting matrix $C = \begin{bmatrix} -1 & 3 & -3 \\ -2 & 6 & -6 \\ -5 & -3 & 1 \end{bmatrix}$ will not be invertible, since the reduced row echelon form of C is not the identity (that is, C is not of full rank). To see why, note that by subtracting 2 times the first row from the second row we will obtain a row of all zeros. Therefore, rref(C) cannot possibly be the identity and C^{-1} does not exist.

2. Let $A = \begin{bmatrix} 4 & 3 & 1 & 2 \\ 1 & 9 & 0 & 2 \\ 8 & 3 & 2 & -2 \\ 4 & 3 & 1 & 1 \end{bmatrix}$.

a) Calculate the determinant of A using a cofactor expansion.

Solution: We expand $\det(A)$ about the third column:

$$\begin{aligned} \det(A) &= 1 \cdot \begin{vmatrix} 1 & 9 & 2 \\ 8 & 3 & -2 \\ 4 & 3 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 4 & 3 & 2 \\ 1 & 9 & 2 \\ 4 & 3 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 4 & 3 & 2 \\ 1 & 9 & 2 \\ 8 & 3 & -2 \end{vmatrix} \\ &= -111 - 66 + 180 \\ &= 3. \end{aligned}$$

b) Recalculate the determinant using row reduction to verify your answer to (a).

Solution: To calculate the determinant, we can put A into upper triangular form using row operations as follows:

$$A = \begin{bmatrix} 4 & 3 & 1 & 2 \\ 1 & 9 & 0 & 2 \\ 8 & 3 & 2 & -2 \\ 4 & 3 & 1 & 1 \end{bmatrix} \xrightarrow[\langle 1\rangle \leftrightarrow \langle 2\rangle]{} \begin{bmatrix} 1 & 9 & 0 & 2 \\ 4 & 3 & 1 & 2 \\ 8 & 3 & 2 & -2 \\ 4 & 3 & 1 & 1 \end{bmatrix} \xrightarrow[\substack{\langle 2\rangle \leftarrow \langle 2\rangle - 4\langle 1\rangle \\ \langle 3\rangle \leftarrow \langle 3\rangle - 8\langle 1\rangle \\ \langle 4\rangle \leftarrow \langle 4\rangle - 4\langle 1\rangle}]{} \begin{bmatrix} 1 & 9 & 0 & 2 \\ 0 & -33 & 1 & -6 \\ 0 & -69 & 2 & -18 \\ 0 & -33 & 1 & -7 \end{bmatrix}$$

$$\xrightarrow[\substack{\langle 3\rangle \leftarrow \langle 3\rangle - 2\langle 2\rangle \\ \langle 4\rangle \leftarrow \langle 4\rangle - \langle 2\rangle}]{} \begin{bmatrix} 1 & 9 & 0 & 2 \\ 0 & -33 & 1 & -6 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & -1 \end{bmatrix} \xrightarrow[\langle 2\rangle \leftrightarrow \langle 3\rangle]{} \begin{bmatrix} 1 & 9 & 0 & 2 \\ 0 & -3 & 0 & -6 \\ 0 & -33 & 1 & -6 \\ 0 & 0 & 0 & -1 \end{bmatrix} \xrightarrow[\langle 3\rangle \leftarrow \langle 3\rangle - 11\langle 2\rangle]{} \begin{bmatrix} 1 & 9 & 0 & 2 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 1 & 60 \\ 0 & 0 & 0 & -1 \end{bmatrix} = U.$$

Therefore, $3 = \det(U) = (-1) \times (-1) \times \det(A)$ so $\det(A) = 3$ as expected.

c) What is the determinant of $-2A$? Why?

Solution: $\det(-2A) = (-2)^4 \det(A) = 16 \cdot 3 = 48$ since A has 4 rows.

3. Prove that if A is an orthogonal matrix (i.e., $A^T = A^{-1}$) then the determinant of A is either 1 or -1.

Solution: Since

$$\det(A) = \det\left(A^T\right) = \det\left(A^{-1}\right) = \frac{1}{\det(A)}$$

we have that $(\det(A))^2 = 1$, so $\det(A) = \pm 1$.

4. Let $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$.

a) Determine the eigenvalues of A.

Solution: The characteristic polynomial is

$$p_A(\lambda) = \det(A - \lambda I) = -\lambda^3 + \lambda = -\lambda(\lambda + 1)(\lambda - 1)$$

so the eigenvalues are $\lambda = 1, -1, 0$.

b) Find a nonsingular matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution: Computing the eigenspaces for each eigenvalue and putting the corresponding fundamental eigenvectors as the columns of a matrix P, we find that $A = PDP^{-1}$ with

$$P = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

c) Compute the determinant of A only using your answer to part (a) (i.e., do not compute the determinant directly.

[Hint: Recall the definition of the characteristic polynomial $p_A(\lambda)$.]

Solution: $\det(A) = p_A(0) = 0$.

5. The parts of the following question are unrelated.

a) Is $\mathcal{V} = \mathbb{R}$ with the usual scalar multiplication, but with addition defined as $\boldsymbol{x} \oplus \boldsymbol{y} = 3(x + y)$ a vector space? Justify your answer.

Solution: No. The operation $\oplus$ is not associative since

$$(\boldsymbol{x} \oplus \boldsymbol{y}) \oplus \boldsymbol{z} = 3(3(x + y) + z) = 9x + 9y + 3z \neq 3x + 9y + 9z = 3(x + 3(y + z)) = \boldsymbol{x} \oplus (\boldsymbol{y} \oplus \boldsymbol{z}).$$

b) Find the zero vector and the additive inverse of the vector space $\mathbb{R}^2$ with operations $[x, y] \oplus [w, z] = [x + w + 3, y + z - 4]$ and $a \odot [x, y] = [ax + 3a - 3, ay - 4a + 4]$.

Solution: $\mathbf{0} = 0 \odot [x, y] = [0x + 3(0) - 3, 0y - 4(0) + 4] = [-3, 4]$ while $-([x, y]) = [-x - 6, -y + 8]$.

c) If $\mathcal{V}$ is a vector space with subspace $\mathcal{W}_1$ and $\mathcal{W}_2$, prove that $\mathcal{W}_1 \cap \mathcal{W}_2$ is also a subspace.

[Hint: Do not forget to show that $\mathcal{W}_1 \cap \mathcal{W}_2$ is nonempty!]

Solution: Since the subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ both contain the zero vector, $\mathbf{0} \in \mathcal{W}_1 \cap \mathcal{W}_2$ and $\mathcal{W}_1 \cap \mathcal{W}_2$ is nonempty. Now suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{W}_1 \cap \mathcal{W}_2$ and c is a scalar. Then $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{W}_1$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{W}_2$ so $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{W}_1$ and $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{W}_2$ since $\mathcal{W}_1$ and $\mathcal{W}_2$ are closed under vector addition. Therefore, $\boldsymbol{x} + \boldsymbol{y} \in \mathcal{W}_1 \cap \mathcal{W}_2$ and $\mathcal{W}_1 \cap \mathcal{W}_2$ is closed under vector addition as well. Similarly we find $\mathcal{W}_1 \cap \mathcal{W}_2$ is closed under scalar multiplication, so $\mathcal{W}_1 \cap \mathcal{W}_2$ is a subspace.

d) Prove that all vectors orthogonal to $[2, -3, 1]^T$ forms a subspace $\mathcal{W}$ of $\mathbb{R}^3$.

Solution: Let $\boldsymbol{v} = [2, -3, 1]^T$. Note that $\mathbf{0} \in \mathcal{W}$ since $\mathbf{0} \cdot \boldsymbol{v} = 0$ so $\mathcal{W}$ is nonempty. Now suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{W}$ and c is a scalar. Then $(\boldsymbol{x} + \boldsymbol{y}) \cdot \boldsymbol{v} = (\boldsymbol{x} \cdot \boldsymbol{v}) + (\boldsymbol{y} \cdot \boldsymbol{v}) = 0 + 0 = 0$ and $(c\boldsymbol{x}) \cdot \boldsymbol{v} = c(\boldsymbol{x} \cdot \boldsymbol{v}) = c0 = 0$.