Burgers turbulence, kinetic theory of shock clustering, and complete integrability

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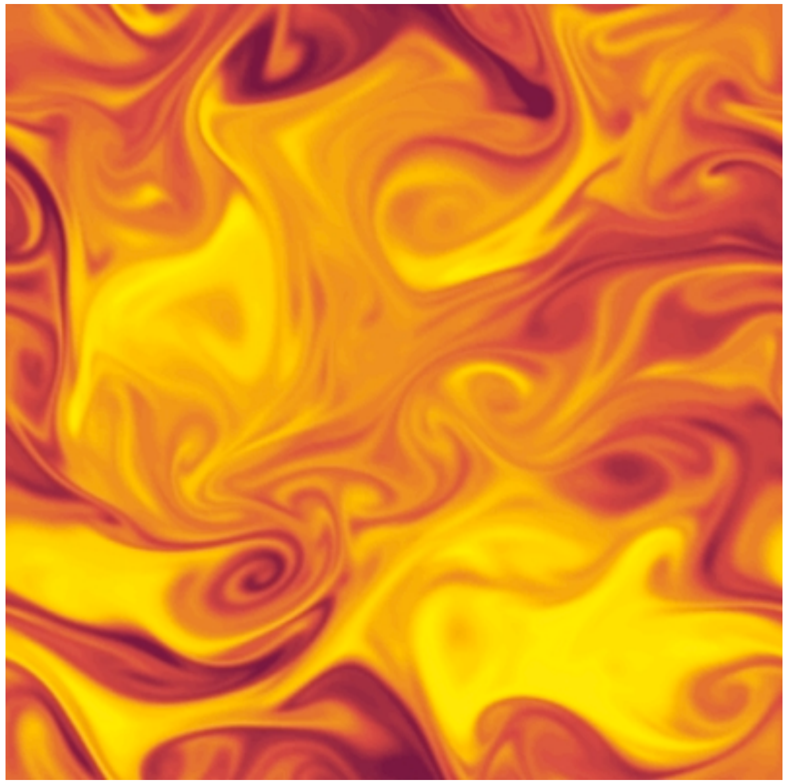
Mostly joint work with Govind Menon (Brown University, USA).

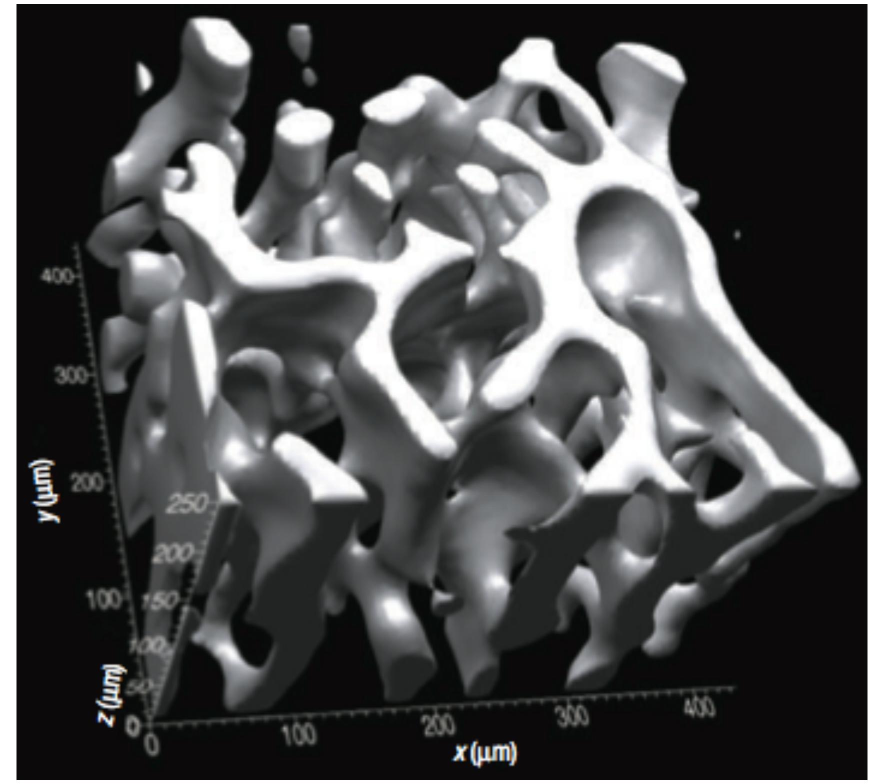
G. Menon & R. Srinivasan, "Kinetic theory and Lax equations for shock clustering and Burgers turbulence," JSP (2010).

G. Menon, "Complete integrability of shock clustering and Burgers turbulence," ArXiv (2011).

R. Srinivasan, "An invariant in shock clustering and Burgers turbulence," ArXiv (2011).



hydrodynamic turbulence



domain coarsening in materials science

Q: Can one construct random fields (stochastic processes) that are solutions to laws of mechanics?

This, of course, is very difficult. One must consider vastly simplified models in order to say something.

Among the simplest of these is Burgers equation with random initial data, known as Burgers turbulence. The earliest results are due to Burgers, who studied the 1-D model with initial data a white noise in space (i.e., complete disorder).

More generally, we will consider random data for 1-D scalar conservation laws with convex flux. As we will see, this leads us very naturally to a problem that lies at the juncture of probability, kinetic theory, and integrable systems.

Main overall result: If

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \qquad x \in \mathbb{R}, t \geq 0$$

with strictly convex flux $f \in C^1$ and $u_0(x)$ a Markov process in x with only downward jumps, *then the* ***law*** *of the entropy solution* $u(x,t)$ *is completely integrable.**

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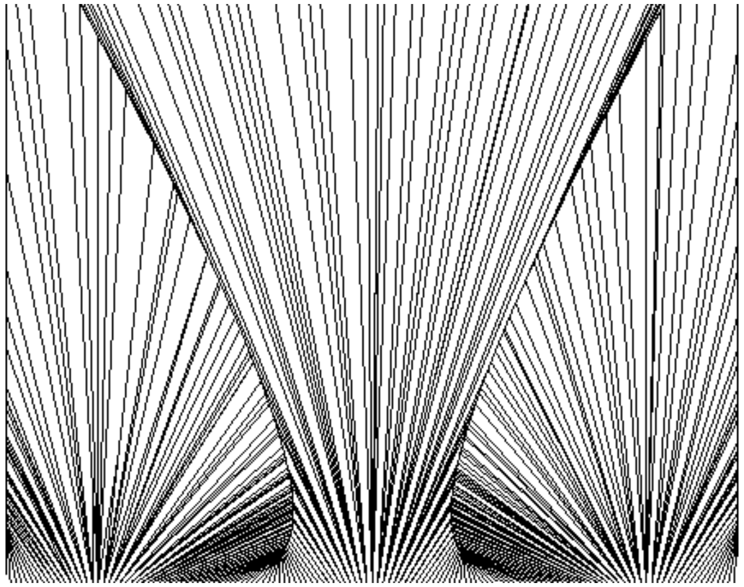
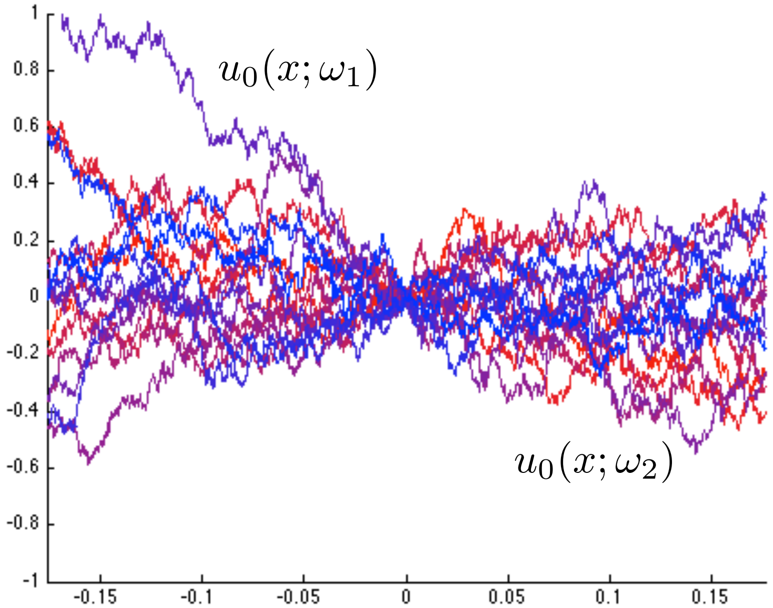
simplest nonlinear law of mechanics + simplest initial random field ⟶ exact solvability





Outline of talk:

$$\partial_t u + \partial_x f(u) = 0$$

$u_0(x)$ Markov $\downarrow$

evolution of random field

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$\longleftrightarrow$

$$\partial_t \mathcal{A} - \partial_x \mathcal{B} = [\mathcal{A}, \mathcal{B}]$$

Lax pair/zero-curvature representation

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evolution of random field

$$\partial_t n = Q(n, n)$$

kinetic equation for shock statistics

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Lax pair/zero-curvature representation

exact solutions; connections to random matrices

Hamiltonian structure; complete integrability; inverse scattering

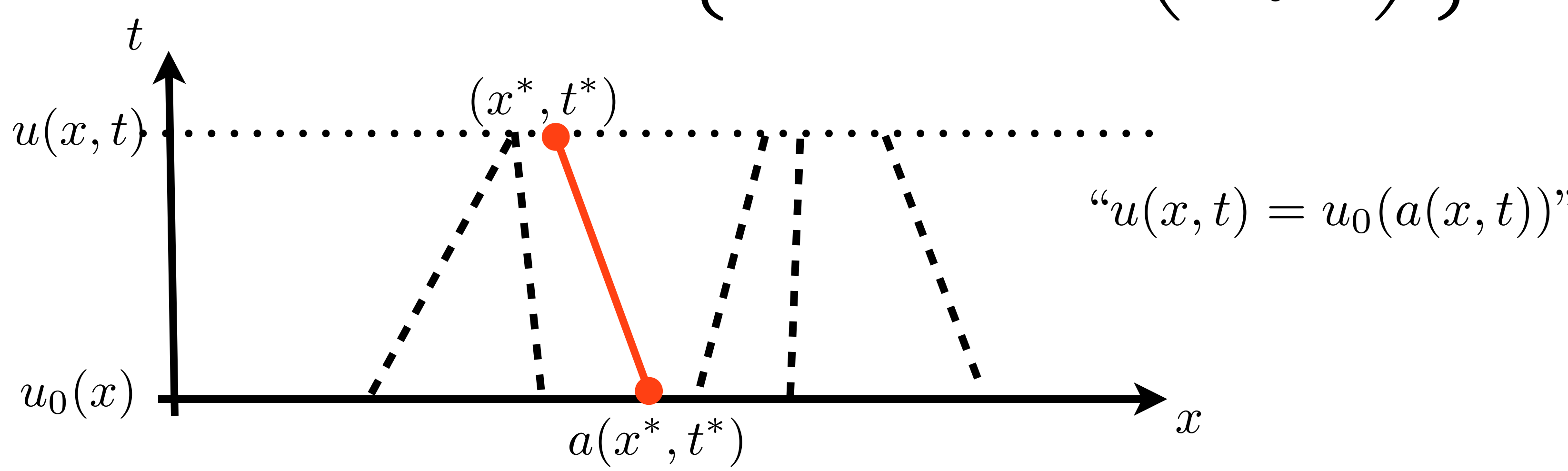
Basics: Entropy solutions to 1-D scalar conservation laws

First, let us review some basic facts about solutions to

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \qquad x \in \mathbb{R}, t \geq 0$$

Classical solutions exist only for short time. The unique *entropy solution* is given by a variational principle (Hopf-Lax formula) in terms of the initial *potential*:

$$a(x,t) = \arg^+ \min_s \left\{ U_0(s) + t f^* \left(\frac{x-s}{t} \right) \right\}$$



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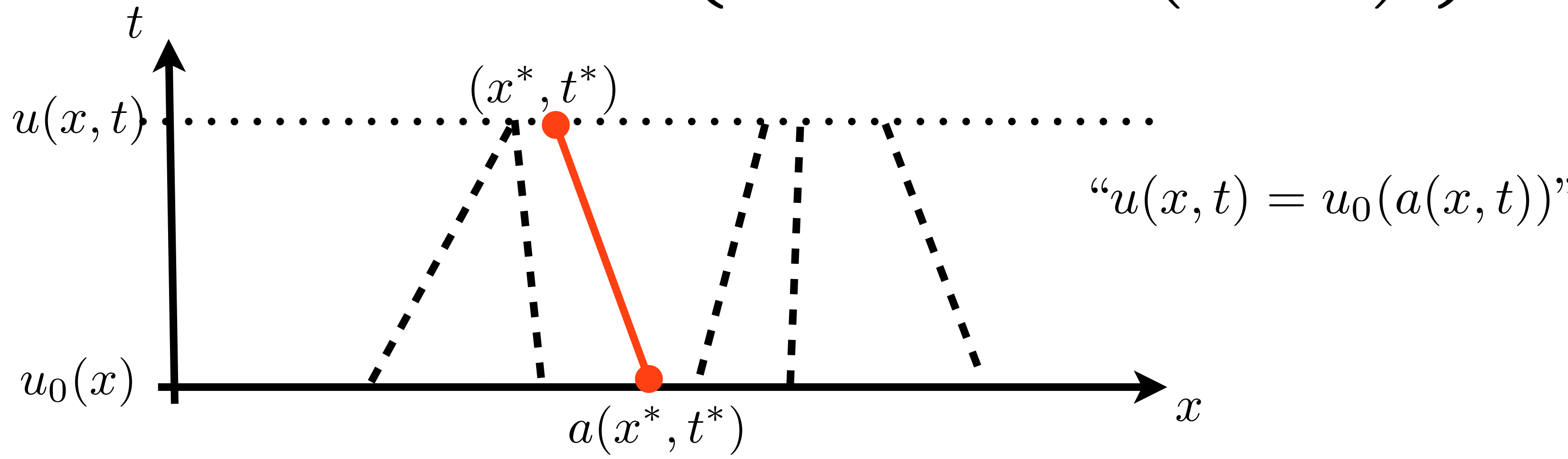
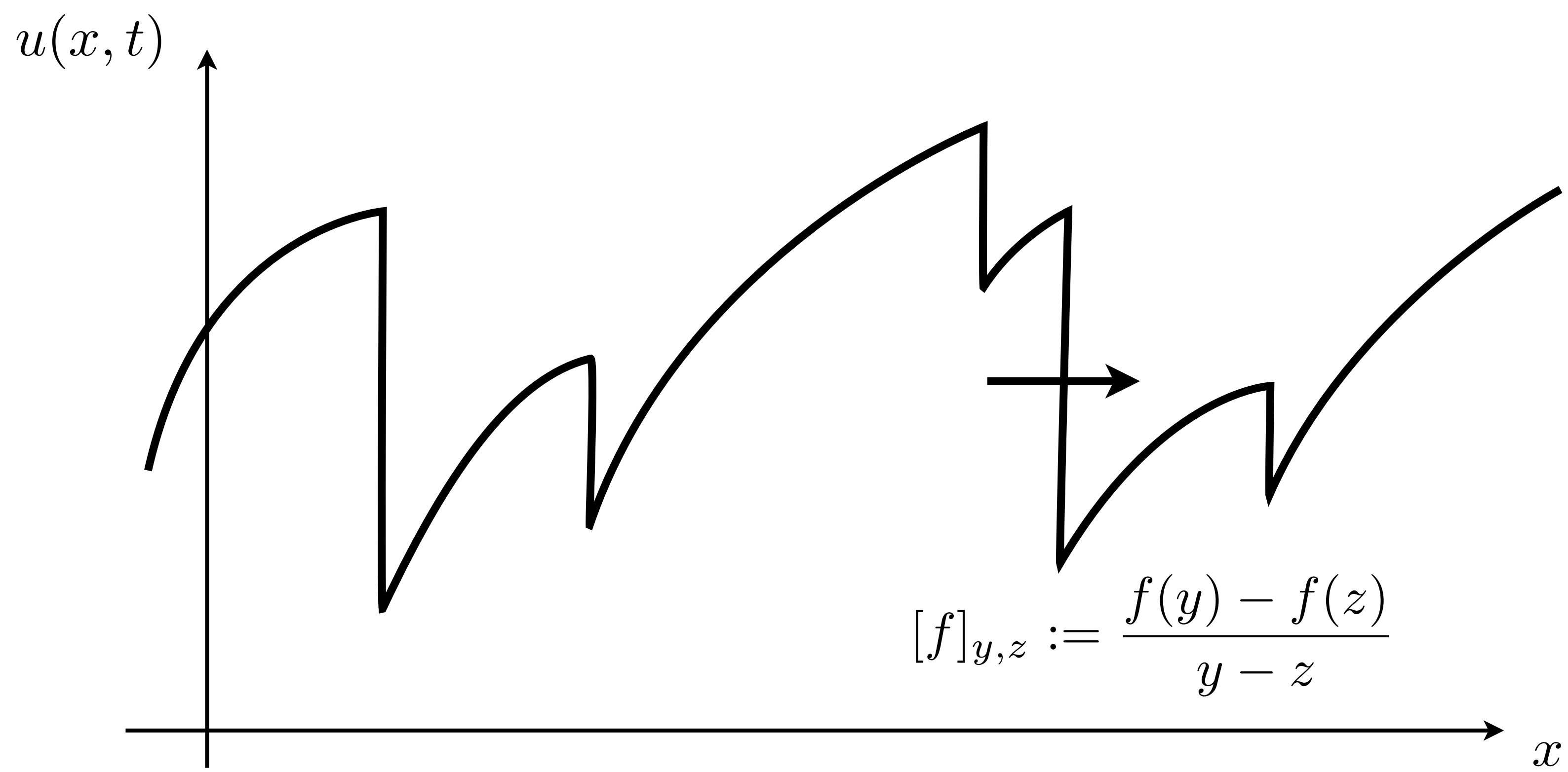


$$f'(u(x,t)) = \frac{x - a(x,t)}{t}$$

For Burgers equation, this is exactly the same procedure as adding a viscous dissipation, using the Cole-Hopf transformation for the potential to obtain the heat equation, and inverting back the corresponding solution.

The evolution regularizes the paths to be of bounded variation for any positive time. Paths consist of rarefaction waves interspersed by downward jumps (shocks).



We will consider $u_0(x)$ to be a stochastic process in x. Since the variational principle is given in terms of the initial *potential*, data can be quite rough (white noise data = BM potential).

1-D scalar conservation laws with random initial data

Our work is greatly motivated by explicit solutions to Burgers with

(I) White noise initial data

[Burgers (‘50s), Groeneboom (’89), Frachebourg-Martin (’00), Avellaneda-E (’95)]

(II) Brownian (Lévy↓) initial data

[Sinai (’92), Carraro-Duchon (’94), Bertoin (’98)]

We return to these later. First, some remarks:

- These data are limits of Markov processes↓. Furthermore, for each fixed t their corresponding solutions are Markov ↓ in x.
- The mechanism of shock clustering holds for any convex flux, not just for Burgers equation.

These *structural* results motivate us to consider

scalar conservation laws with convex flux and Markov ↓ initial data

Markov processes are characterized by transition semigroups $\{\mathcal{Q}_h\}_{h\geq 0}$.
Their generators are given by their action on test functions φ:

$$\mathcal{A}\varphi = \lim_{h\downarrow 0} \frac{1}{h}(\mathcal{Q}_h\varphi - \varphi)$$

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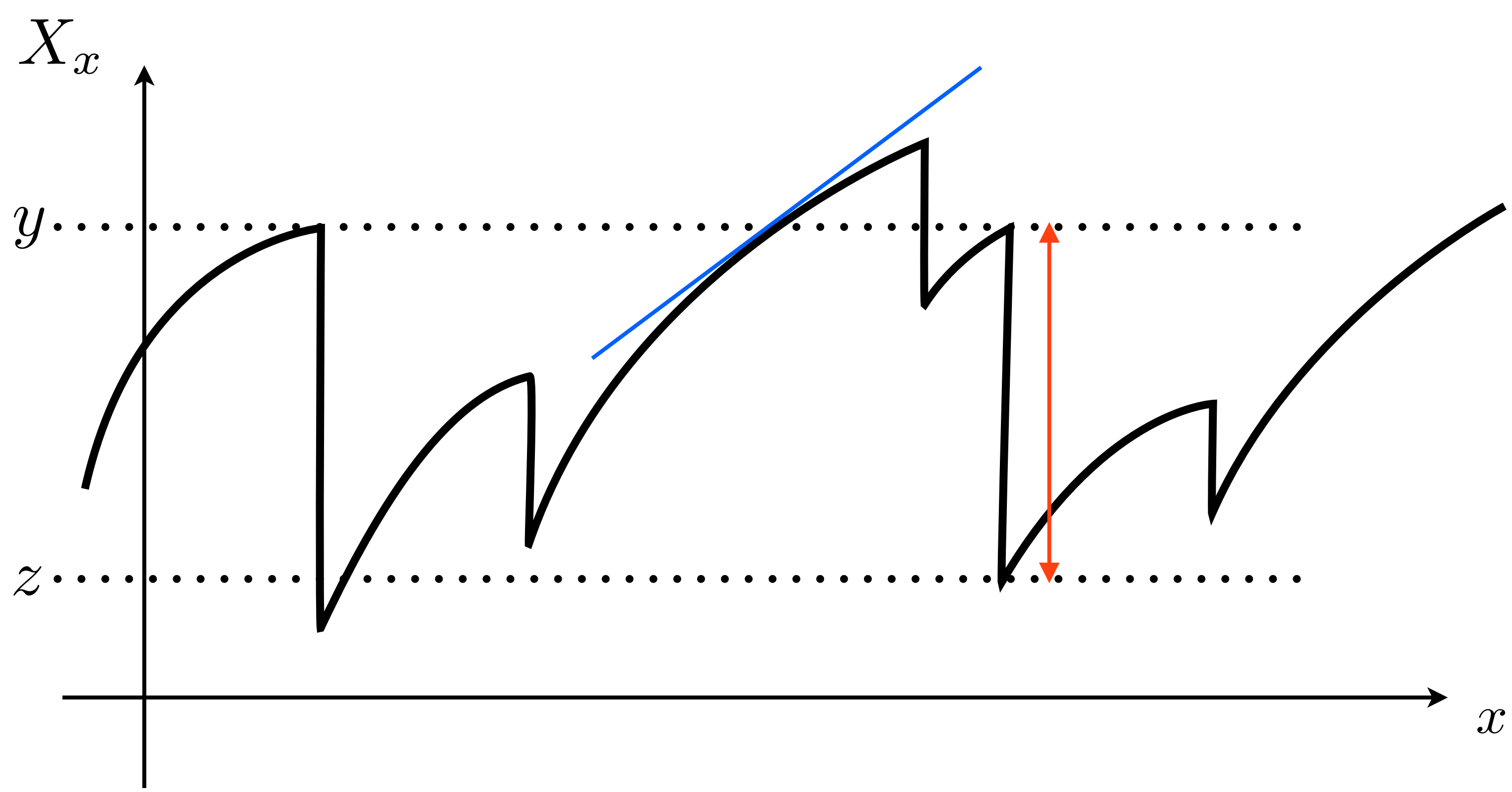
For stationary processes with BV paths and only downward jumps,

$$\mathcal{A}\varphi(y) = \underbrace{b(y)}_{\text{drift}}\,\varphi'(y) + \int_{z<y} \underbrace{n(y,dz)}_{\text{jumps}}(\varphi(z)-\varphi(y))$$



Theorem [Menon-Srinivasan (2010)]:
If $u_0(x)$ is strong Markov with only downward jumps (spectral negativity), then for fixed $t > 0$ and *any* strictly convex $f \in C^1$, the entropy solution $u(x,t)$ is a Markov process in x.

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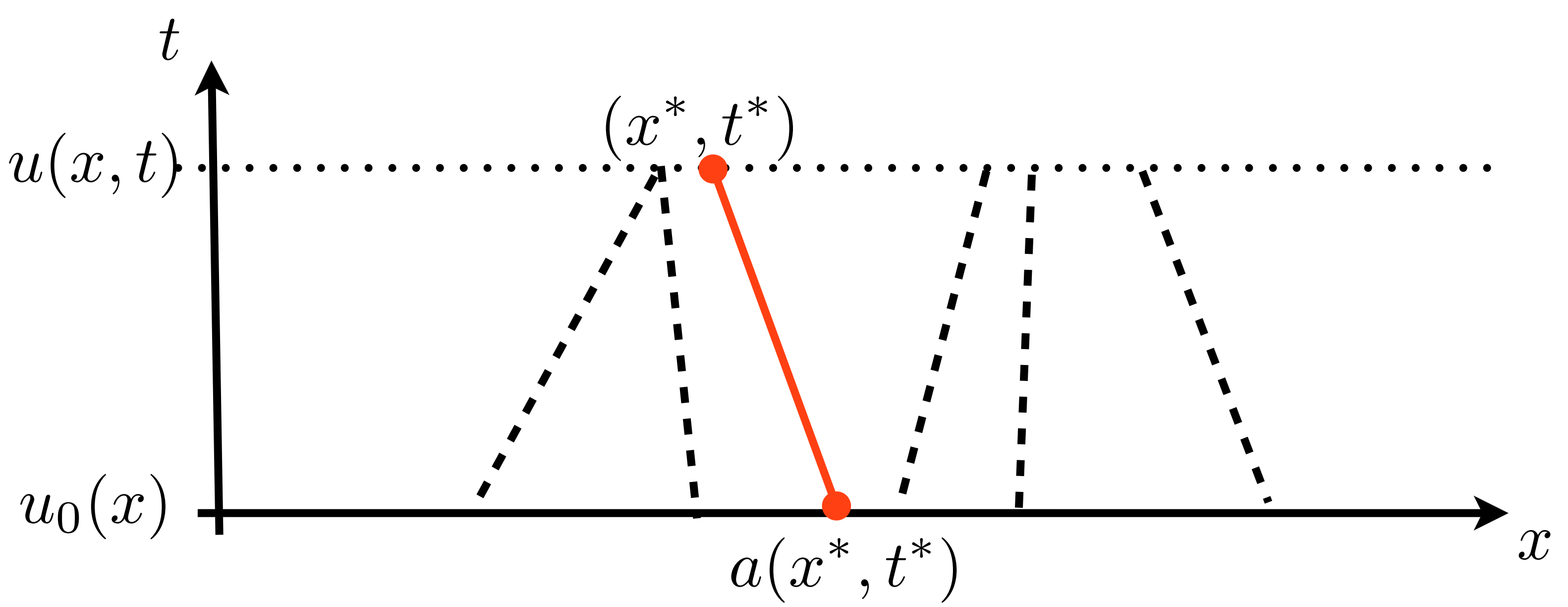
The same result holds for initial data which are the limit of spectrally negative Markov processes (for example, white noise).

Their motivation and proofs rely on previous work by Bertoin ('98), Groeneboom ('89), Chabanol-Duchon ('04) and others. As we will see, the generalization to all convex fluxes plays an important role.

Proof by picture:



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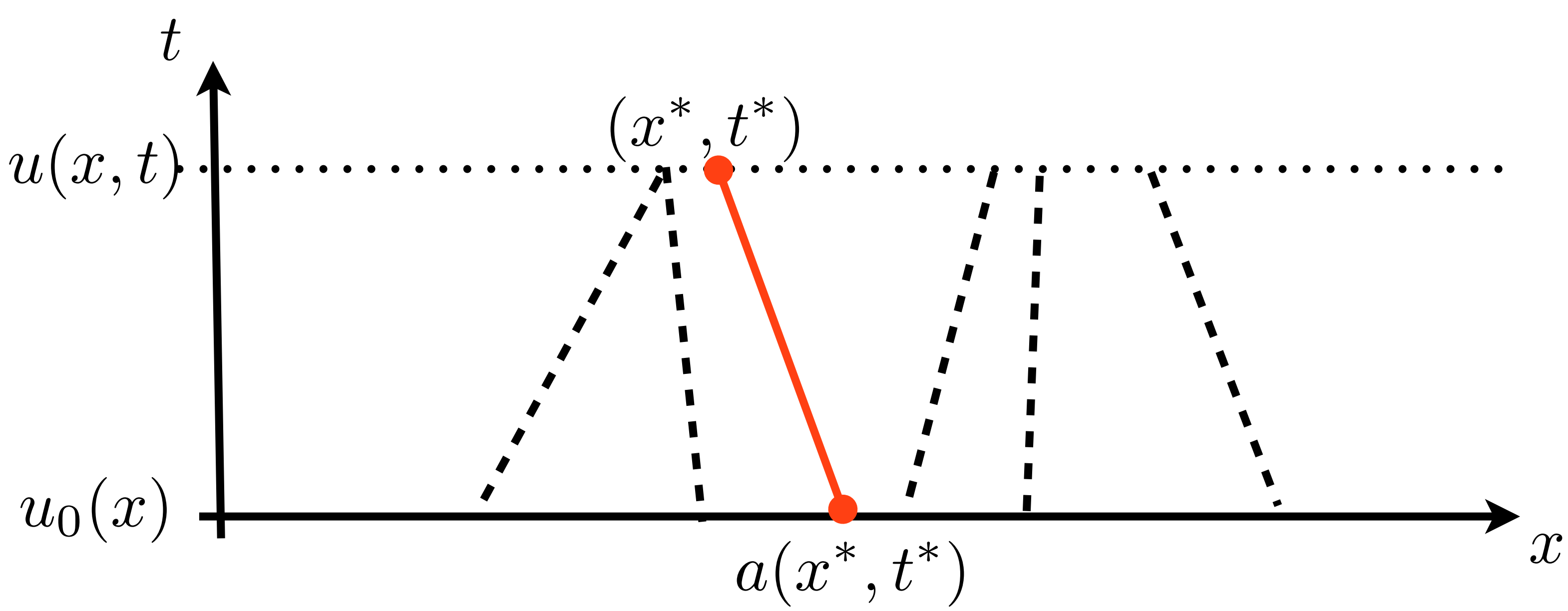

The Markov property holds at the random `time' $a(x^*, t^*)$ as it is the last-passage of a strong Markov process (Hopf-Lax functional) at its minimum.

So for each fixed time t, we have a generator of a process in x:

$$\mathcal{A}(x,t)\varphi(y) = b(y;x,t)\varphi'(y) + \int_{z<y} n(y,dz;x,t)(\varphi(z) - \varphi(y))$$

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$$[f]_{y,z} := \frac{f(y) - f(z)}{y - z}$$

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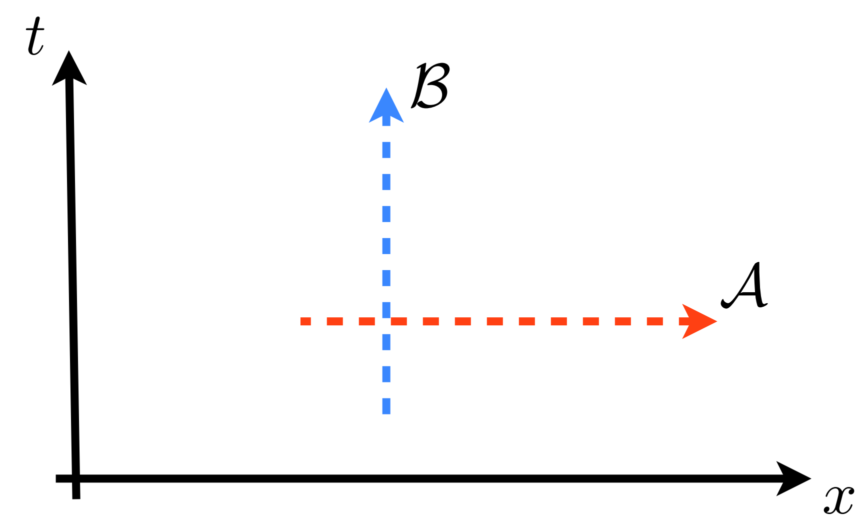

Derive "generator" in time by Itô's formula for jump processes:

$$\mathcal{B}(x,t)\varphi(y) = \quad -b(y;x,t)f'(y)\varphi'(y) - \int_{z<y} n(y,dz;x,t)[f]_{y,z}(\varphi(z)-\varphi(y))$$

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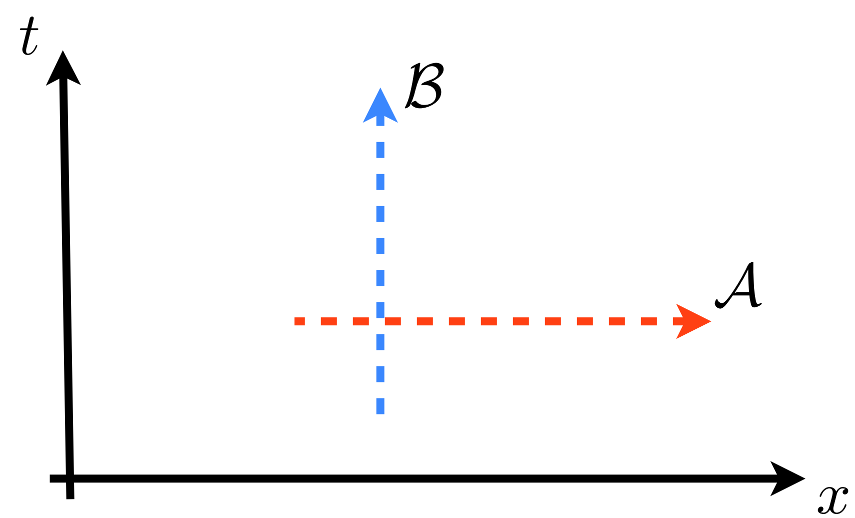

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$$\partial_t \mathcal{A} - \partial_x \mathcal{B} = [\mathcal{A},\mathcal{B}]$$

This is seen by considering backward Kolmogorov equations for the solution semigroups in x and t:

$$\partial_x \varphi + \mathcal{A}(x,t)\varphi = 0$$
$$\partial_t \varphi + \mathcal{B}(x,t)\varphi = 0$$

The zero-curvature equation $\partial_t \mathcal{A} - \partial_x \mathcal{B} = [\mathcal{A}, \mathcal{B}]$ appears when we enforce compatibility ($\partial_{xt}\varphi = \partial_{tx}\varphi$).

If the initial data is a stationary Markov process then $\mathcal{A}$ and $\mathcal{B}$ do not depend on x then we have the Lax equation

$$\boxed{\partial_t \mathcal{A} = [\mathcal{A}, \mathcal{B}]}$$

We have three other independent derivations using:

(i) a kinetic formulation by considering evolution of a single shock [as in Menon-Pego ('07)]

(ii) Vol'pert's BV calculus and Markov property [as in E-Vanden-Eijnden ('00)]

(iii) Hopf functional equation for the Fourier transform of the law of the solution [as in Chabanol-Duchon ('04)]

A rigorous proof is still missing. This requires a closure property for Feller processes (Markov = measurability, Feller = regularity).

$$\mathcal{A}(x,t)\varphi(y) = b(y;x,t)\varphi'(y) + \int_{z<y} n(y,dz;x,t)(\varphi(z)-\varphi(y))$$

The Lax/ZC equation is an evolution equation for the drift coefficient and the jump measure of $\mathcal{A}(x,t)$. What is this equation explicitly?

Suppose jump measure has density $n(y, z, t)$. Then

$$\partial_t \mathcal{A} = [\mathcal{A}, \mathcal{B}] \quad \Longleftrightarrow$$

Vlasov-Boltzmann equation
for shock statistics

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Vlasov-Boltzmann equation for shock statistics

Drift: $\partial_t b(y,t) = -f''(y) b^2(y,t)$

Shocks:
$$\begin{aligned} \partial_t n(y,z,t) \quad &+ \quad \partial_y(nV_y) + \partial_z(nV_z) \\ &= \quad Q(n,n) + n(([f]_{y,z} - f'(y))\partial_y b - bf''(y)) \end{aligned}$$

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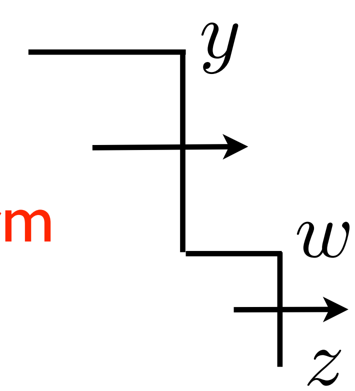
$$V_y(y,z,t) = ([f]_{y,z} - f'(y))\, b(y,t), \quad V_z(y,z,t) = ([f]_{y,z} - f'(z))\, b(z,t)$$

$$\begin{aligned} Q(n,n)(y,z,t) \quad &= \int_z^y ([f]_{y,w} - [f]_{w,z})\, n(y,w,t) n(w,z,t)\, dw \\ &- \int_{-\infty}^z ([f]_{y,z} - [f]_{z,w})\, n(y,z,t) n(z,w,t)\, dw \\ &- \int_{-\infty}^y ([f]_{y,w} - [f]_{y,z})\, n(y,z,t) n(y,w,t)\, dw. \end{aligned}$$

This is easily interpreted for Burgers turbulence and initial data with linear drift. In particular, mean-field theory is exact:



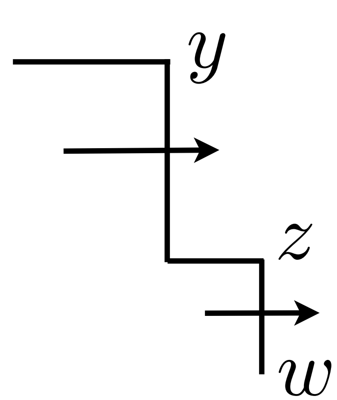
free streaming

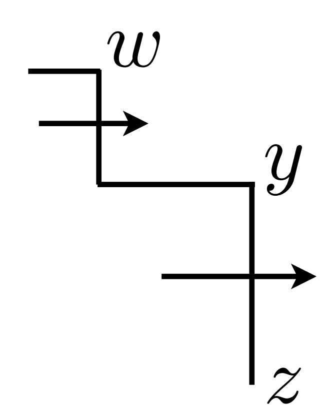
birth term

$$\partial_t n(y,z) = \quad t^{-1}\left(\frac{y-z}{2}\right)(\partial_y n - \partial_z n) + \left(\frac{y-z}{2}\right)\int_{\mathbb{R}} n(y,w)n(w,z)dw$$
$$-n(y,z)\int_{\mathbb{R}} n(z,w)dw\left(\frac{y-w}{2}\right) - n(y,z)\int_{\mathbb{R}} n(y,w)\left(\frac{w-z}{2}\right)dw$$

death from below

death from above





Equivalence between Lax equation and kinetic equation obtained using that operators of the form

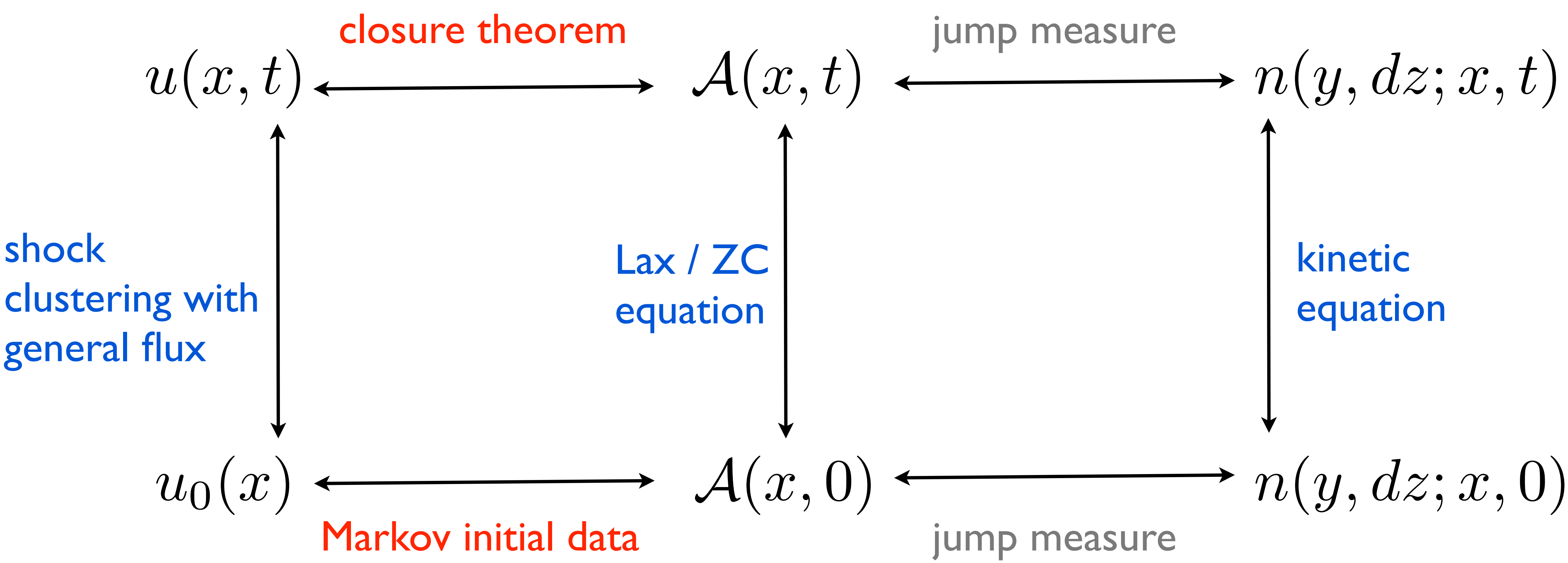
$$\mathcal{A}\varphi(y) = b(y)\varphi'(y) + \int_{z<y} n(y,z)(\varphi(z) - \varphi(y))dz$$

formally constitute a Lie algebra with bracket given by commutator. In particular, $[\mathcal{A}, \mathcal{B}]$ also takes the form given above.

Q: To elucidate a Hamiltonian structure, is there an ∞-dim Lie group generated by this algebra? More on this later...

To summarize:

$$\mathcal{A}(x,t)\varphi(y) = b(y;x,t)\varphi'(y) + \int_{z<y} n(y,dz;x,t)(\varphi(z)-\varphi(y))$$



As mentioned earlier, there are exact, self-similar solutions for Burgers equation with initial data a white noise or a Brownian motion (Lévy process).

Exact solution (I): Burgers equation with white noise initial data

- P. Groeneboom, Brownian motion with a parabolic drift and Airy functions, Prob. Th. Rel. Fields, 81 (1989).
- L. Frachebourg & P. Martin, Exact statistical properties of the Burgers equation, J. Fluid Mech., 417 (2000).

Initial potential is a 2-sided BM. Exact solution derived by Groeneboom (1989) (through purely probabilistic methods!)

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Solution process is Markov in space. Statistics are self-similar, so we need only consider one-point and transition densities at time 1. First, the one-point density:

$$p(y) = P(u(x,1) \in dy)/dy = J(y)J(-y) \sim e^{-\frac{2}{3}|y|^3}, \qquad |y| \to \infty$$

$$j(q) = \int_0^\infty e^{-qy} J(y)dy = \frac{1}{\mathrm{Ai}(q)}$$

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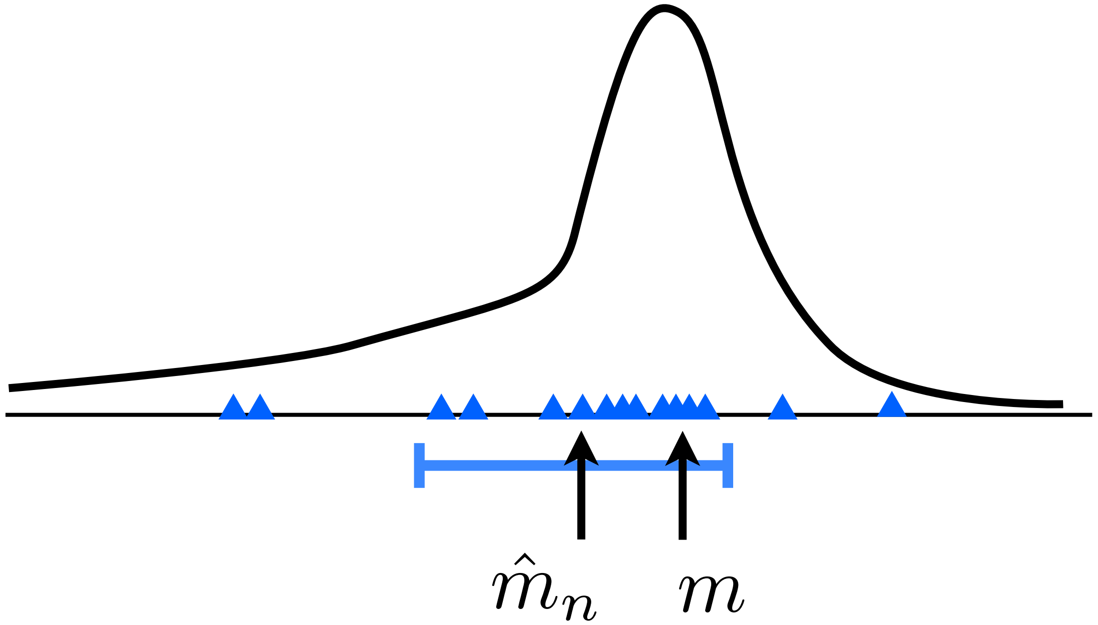
Chernoff (1964): Estimation of the mode using a non-smooth kernel

$$\lim_{n\to\infty} \frac{\hat{m}_n - m}{cn^{1/3}} \sim p(u)$$

"cube-root asymptotics"



Next, the transition probabilities given by components of solution generator:

$$\mathcal{A}(t)\varphi(y) = \frac{1}{t}\varphi'(y) + \int_{z<y} \frac{1}{t^{1/3}} n_*(yt^{1/3}, zt^{1/3})(\varphi(z) - \varphi(y))$$

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Not originally presented this way.

Groeneboom's solution satisfies the Lax equation. This is checked by using the following identities for $l = (\log j)'$ and k:

$$l' = -q + l^2$$
$$k' = -2(1 - lk)$$
$$k''' = 3(k^2)' + 4qk' + 2k$$

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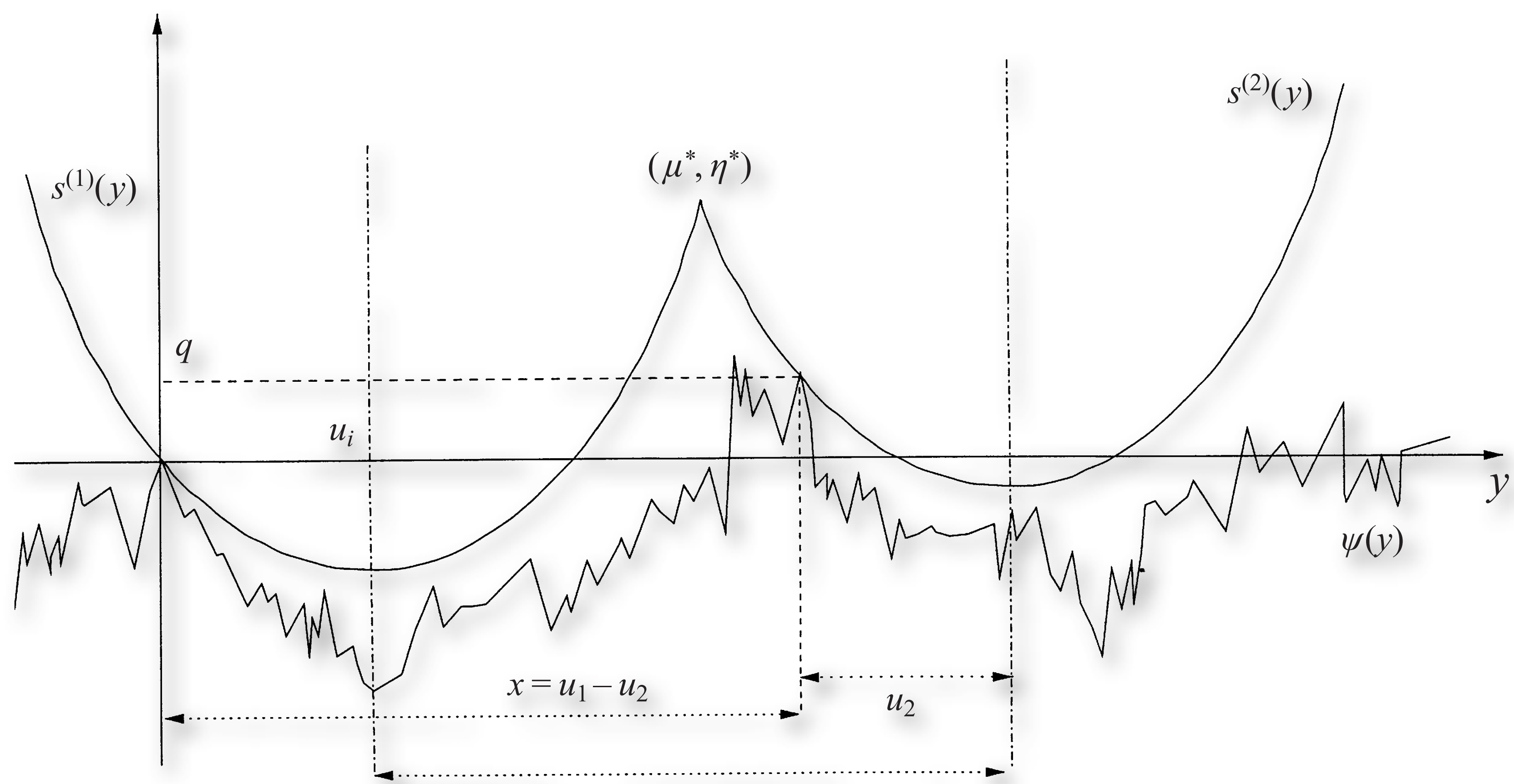
$l = (\log j)'$ is an Airy solution to Painlevé-II w/parameter 1/2:

$$w'' = 2w^3 + 2wq + \frac{1}{2}$$

Additionally, we can express Groeneboom's solution in terms of the Airy kernel by using the resolvent of the Markov semigroup.

A comment:
Groeneboom's derivation is a study of Brownian excursions on parabolic boundaries (Hopf-Lax functional). Airy functions arise upon considering the diffusion process after flattening the boundary (Girsanov's theorem). It remains to be seen how the solution can be rederived using techniques from integrable systems.



Exact solution (II): Burgers equation with Brownian initial data

- L. Carraro & J. Duchon, C.R. Acad. Sc. Paris Math., 319 (1994); Ann. IHP Anal. Nonlineaire, 15 (1998).
- J. Bertoin, The inviscid Burgers equation with Brownian initial velocity, Comm. Math. Phys., 193 (1998).

Lévy processes are simply Markov processes with independent increments. Their transition probabilities are invariant in state space (e.g., BM started at 1 is equivalent to BM started at 0, modulo upward shift).

Closure: Burgers equation preserves the class of Lévy processes$\downarrow$.

Such processes are characterized by their Laplace exponent $\psi(q)$ (i.e., symbol of generator):

$$\mathcal{A}e^{qy} = \psi(q)e^{qy}, \qquad q > 0$$

$$\mathcal{A}(t)e^{qy} = \psi(q,t)e^{qy}$$

$$[\mathcal{A}(t),\mathcal{B}(t)]e^{qy} = -\psi(q,t)\partial_q\psi(q,t)e^{qy}$$

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$$\partial_t\mathcal{A} = [\mathcal{A},\mathcal{B}] \qquad \Longleftrightarrow \qquad \partial_t\psi + \psi\partial_q\psi = 0$$

Laplace exponent of solution process also satisfies Burgers equation! This had been obtained in the past [Bertoin, Duchon et. al] and the Lax equation gives the same result.

BM initial data gives a self-similar solution:

$$\psi_0(q) = q^2$$

$$\psi(q,t) = t^{-2}\psi_*(qt), \qquad \psi_*(q) = q + \frac{1}{2} - \sqrt{q + \frac{1}{4}}$$

Connections to random matrices and Wigner semicircle law

BM initial data for Burgers turbulence connected to Wigner semicircle law via Dyson's BM for eigenvalues of random matrices...

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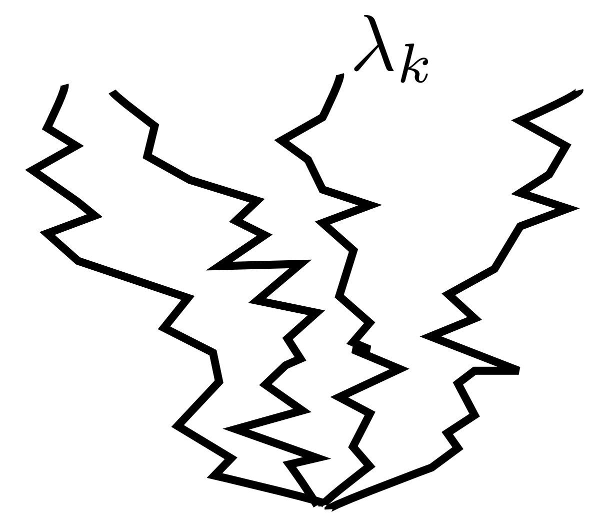
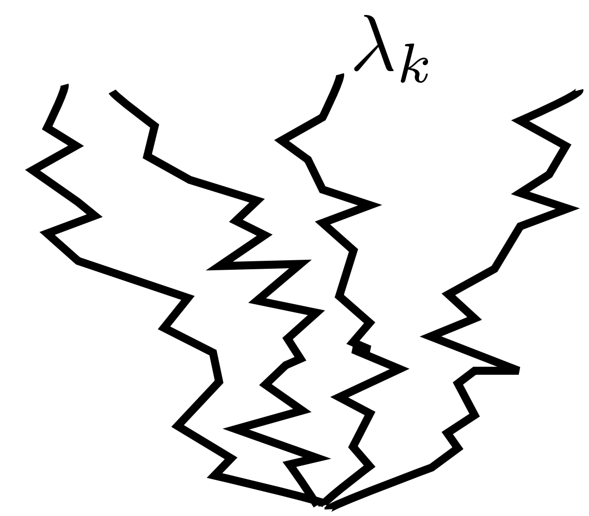
$$d\lambda_k = \sum_{j \neq k} \frac{dt}{\lambda_k - \lambda_j} + \sqrt{\frac{2}{\beta}}\, dB_k, \quad 1 \leq k \leq n$$



BM initial data for Burgers turbulence connected to Wigner semicircle law via Dyson's BM for eigenvalues of random matrices...

$$d\lambda_k = \sum_{j \neq k} \frac{dt}{\lambda_k - \lambda_j} + \sqrt{\frac{2}{\beta}} dB_k, \quad 1 \leq k \leq n$$



Rescale $x_k = \lambda_k / \sqrt{n}$, apply Cauchy transform:

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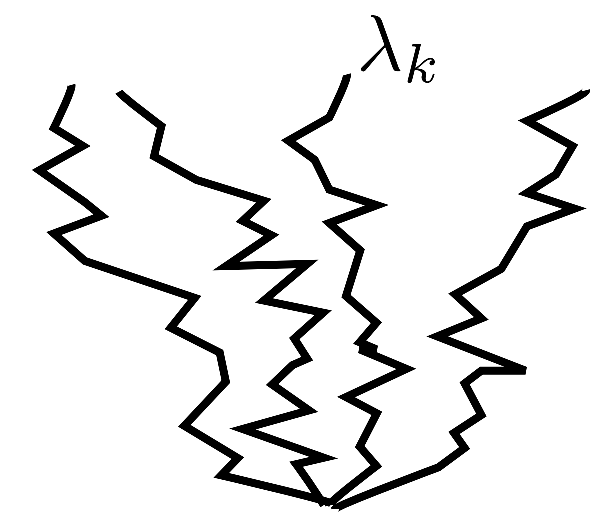

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In large n limit, $F_n(t,dx) \longrightarrow \mu(t,dx)$ and g satisfies Burgers eqn.:

$$\partial_t g + g \partial_z g = 0$$
$$g(z,0) = 1/z$$

Self-similar solution is transform of Wigner semicircle law:

$$g(z,t) = \frac{1}{\sqrt{t}} g_* \left(\frac{z}{\sqrt{t}} \right) \qquad g_*(z) = \frac{1}{2}(z - \sqrt{z^2 - 4}), \quad |z| \geq 2$$

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SS soln's to BM-Burgers turbulence and Dyson's BM related through their respective integral transforms (Laplace transform of law vs. Cauchy transform of empirical measure):

$$\frac{\psi_*(q)}{q} = g_*(z), \quad z = 2 + \frac{1}{q}$$

BM in Burgers turbulence

Laplace transform

Cauchy transform

Dyson's BM for random matrices

SS solution to Burgers eq'n

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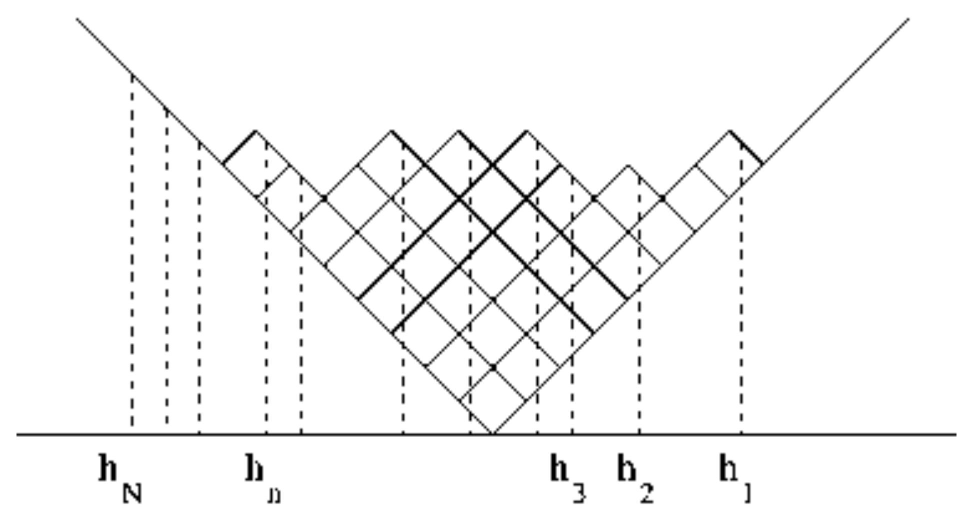

Plancherel growth

Representation theory of infinite symmetric group (Kerov, 2000)

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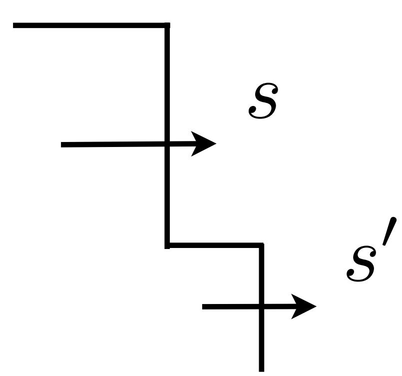
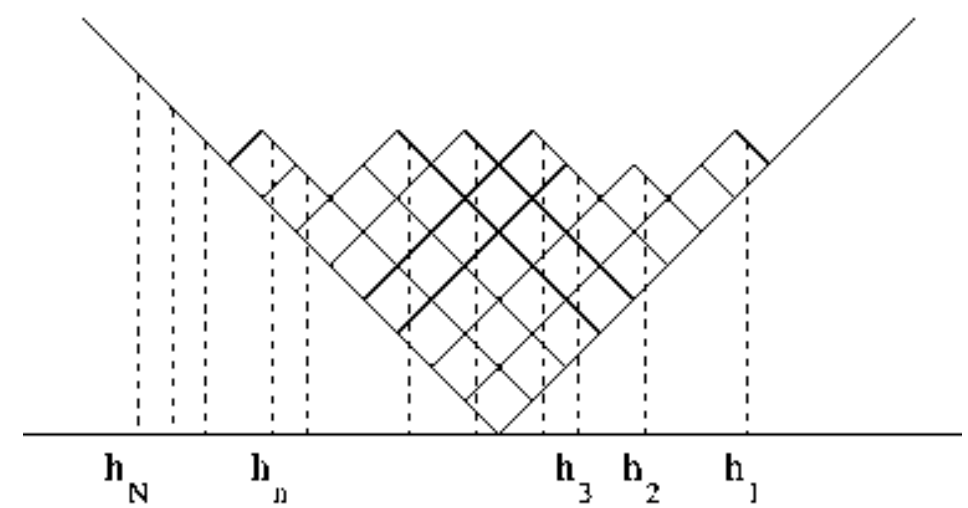




"Smoluchowski growth"

Plancherel growth

Representation theory of infinite symmetric group (Kerov, 2000)

Hamiltonian structure and complete integrability

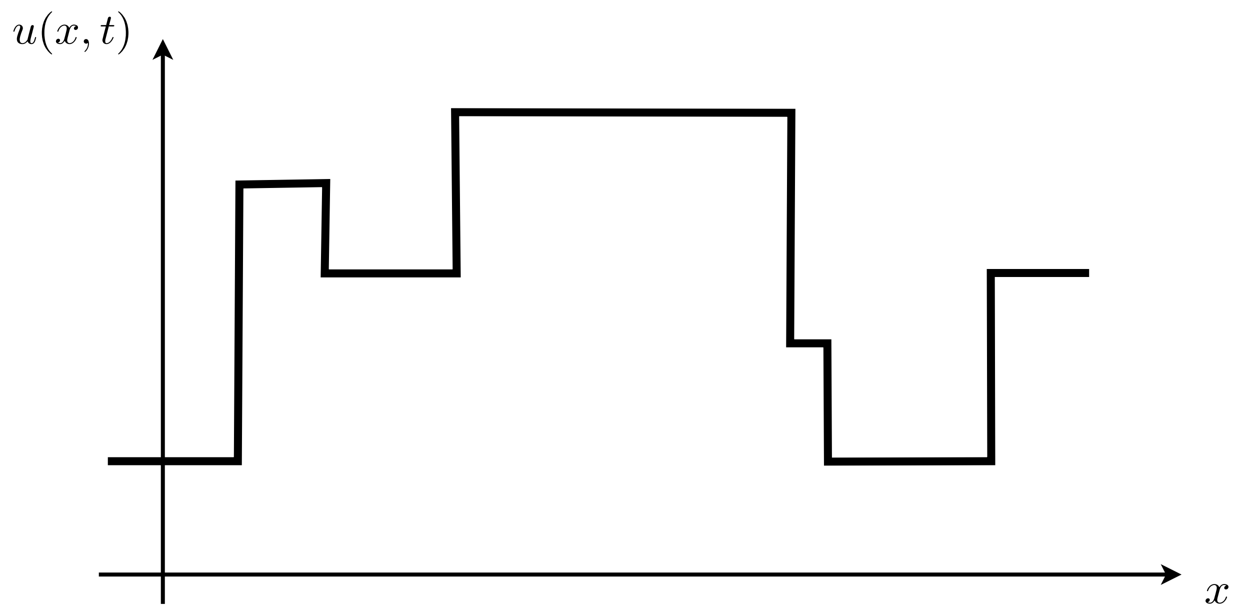
G. Menon, "Complete integrability of shock clustering and Burgers turbulence," ArXiv (2011)

Main result:
When discretized, the Lax equation yields a Hamiltonian flow given by a principle of least action on `Markov groups.' The deeper meaning/implication seems to be:

(i) The space of Markov (Feller) processes ↓ with BV paths admits a natural symplectic structure

(ii) Every convex flux $f \in C^1$ generates a Hamiltonian flow with respect to this structure, and these flows commute

Finite-dimensional projections of the Lax equation are natural as they describe the evolution of a generator (now a matrix) of a continuous-time Markov chain on a finite set of states $y_1 < y_2 < \cdots < y_N$.



The generator is now a matrix with positive entries on the off-diagonal and row sums equal to 0. If we relax the positivity constraint, this yields a Lie algebra $\mathfrak{m}(N)$ (call `Markov algebra').

This generates a `Markov group' $\mathfrak{M}(N)$.

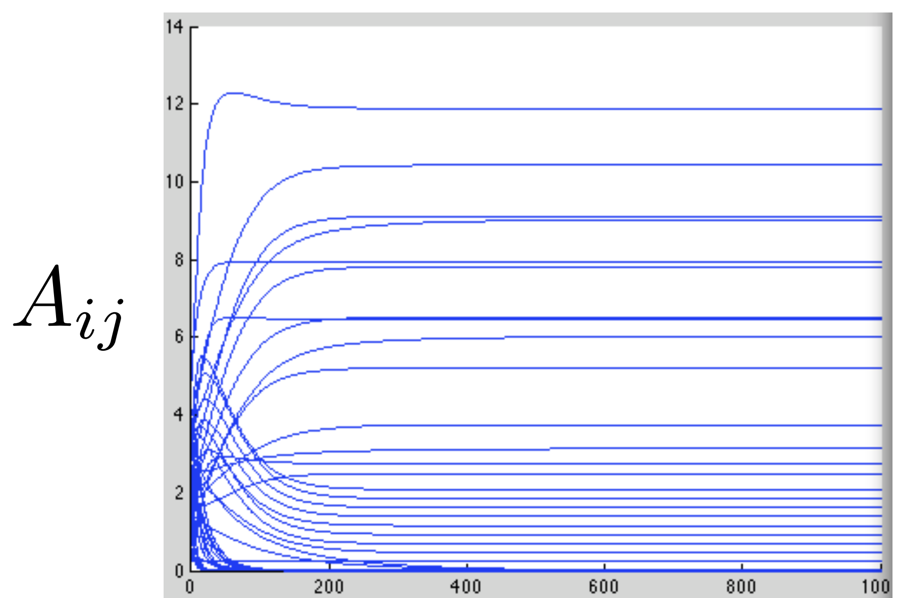
$$\dot{A} = [A, B]$$

$$B_{ij} = F_{ij} A_{ij}, \quad i \neq j \qquad F_{ij} = -\frac{f(y_i) - f(y_j)}{y_i - y_j}$$

Analogous to geodesic flow on SO(N) (Euler's equation for rigid body motion)! In that case, the multiplier F_{ij} is positive and defines a metric (and quadratic action) on SO(N). Geodesic flow is Hamiltonian with respect to this group structure.

In our model, F_{ij} instead yields a principle of least action on the Markov group $\mathfrak{M}(N)$. The flow can be shown to be Hamiltonian by an appropriate splitting at the level of Lie algebras.

Most importantly from probabilistic perspective, this flow is invariant on the space of lower triangular Markov matrices that are positive on the off diagonal (generators of Markov processes with only downward jumps).



Therefore, the original model is Hamiltonian (at least at the level of natural discretizations).

The Lax equation does not itself imply integrability (only gives N, not N^2 invariants). However, it admits a spectral parameter as in Manakov (1976) for geodesic flow on SO(n):

$$[\mathcal{A}, \mathcal{N}] - [\mathcal{M}, \mathcal{B}] = 0$$

$$\mathcal{M}\varphi(y) = y\varphi(y), \qquad \mathcal{N}\varphi(y) = f(y)\varphi(y)$$

That is,

$$\partial_t (\mathcal{A} - \mu\mathcal{M}) = [\mathcal{A} - \mu\mathcal{M}, \mathcal{B} + \mu\mathcal{N}], \qquad \mu \in \mathbb{C}.$$

For finite-dim. systems this gives an invariant spectral curve (and a Riemann-Hilbert problem via a matrix factorization):

$$\Gamma = \{(\lambda, \mu) \in \mathbb{C}^2 | \det(\mathcal{A} - \lambda \mathrm{Id} - \mu\mathcal{M}) = 0\}$$

Using existing machinery and this `Markov group' structure yields complete integrability of the flow.

Furthermore, the ZC equation can be solved by inverse scattering. In particular, the ZC equation is actually a variant of the n-wave model of Zakharov and Manakov (1973) from nonlinear optics!

Summary:

$$\partial_t u + \partial_x f(u) = 0$$

$u_0(x)$ Markov $\downarrow$

evolution of random field

$$\partial_t n = Q(n, n)$$

kinetic equation for shock statistics

$$\partial_t \mathcal{A} - \partial_x \mathcal{B} = [\mathcal{A}, \mathcal{B}]$$

Lax pair/zero-curvature representation

exact solutions; connections to random matrices

Hamiltonian structure; complete integrability; inverse scattering

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Thanks for your attention!

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