18/06/13

Atiyah-Singer Index Theorem Seminar.

Spin groups I (Aaron Fenyes).

Spin groups (dim ≥ 3): Motivation, Construction, and... there's an issue with representations... :(.

Motivation: Quantum Mechanics.

In QM, every system is represented by a complex complete inner product space, i.e.,

$$\text{physical system} \longleftrightarrow \text{Hilbert space } \mathcal{H}$$

More physical/mathematical analogies (or correspondences):

logical props. about system $\longleftrightarrow$ closed subspaces of $\mathcal{H}$

"maximal" propositions (e.g. atomic props) "pure state" $\longleftrightarrow$ 1 dim subspace of $\mathcal{H}$; elt of $\mathbb{P}\mathcal{H}$.

Take the projectivisation $\mathcal{H} \xrightarrow{\pi} \mathbb{P}\mathcal{H}$, and define

$$P : (\mathbb{P}\mathcal{H})^2 \longrightarrow [0,1] \text{ by } P(\pi x, \pi y) = \frac{|\langle x, y\rangle|^2}{\|x\|^2 \cdot \|y\|^2}.$$

NOTE: $P(\psi,\phi) = \cos^2(d(\psi,\phi))$

$\llcorner$ Fubini-Study metric.

Let $A \subset \mathcal{H}$ be a proposition (closed subspace), and let $\psi \in \mathbb{P}\mathcal{H}$. Then we interpret this angle P probabilistically (to talk about our actual physical systems/make prob. predictions) via:

$$\text{Prob}(A|\psi) = \min_{a \in A} P(a,\psi) \qquad \text{"Born rule"}$$

All this together gives a self-consistent framework for describing & predicting physical systems.

Transformations:

If we transform our system physically, it should preserve some aspect of our system and thus some kind of mathematical data. The <u>weakest</u> notion of preservation that makes sense is that the transformation preserves P, the probability angle.

transformation of system from type $\mathcal{H}$ to type $\mathcal{H}'$ $\longrightarrow$ WEAKEST NOTION map $\mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}'$ that preserves P

$\llcorner$ equivalently, a Fubini-Study isometry

Theorem (Wigner):

$$\begin{tikzcd} \mathcal{H} \arrow[r, dashed, "\substack{\text{linear or antilin.}\\ \text{isometry,}\\ \text{unique up to}\\ \text{mult by } U(\mathbb{C})}"] \arrow[d, "\pi"'] & \mathcal{H}' \arrow[d, "\pi'"] \\ \mathbb{P}\mathcal{H} \arrow[r, "\substack{\text{Fubini-Study}\\ \text{isometry}}"'] & \mathbb{P}\mathcal{H}' \end{tikzcd}$$

Remark: $T: \mathcal{H} \to \mathcal{H}'$ is antilinear if

$$T(x+y) = Tx + Ty$$
$$T(\lambda x) = \bar{\lambda} Tx$$

An antilinear map is an isometry if $\langle Tx, Ty \rangle = \langle y, x \rangle$.

Symmetry Groups:

Notation/convention: $U(1) = U(\mathbb{C}) = S^1 \subset \mathbb{C}$.

Consider a physical system $\mathcal{H}$ with a group $G \circlearrowright \mathbb{P}\mathcal{H}$ by isometries.

By Wigner, for each $g \in G$, there exists a unique (up to $U(\mathbb{C})$) element $\rho(g)$ of $U(\mathcal{H})$ or $\bar{U}(\mathcal{H})$ that lifts g.

Observation ①: If G is a connected Lie group, $\rho(g)$ must always be unitary.

Proof:

G connected, so all elt$^{\underline{s}}$ can be written $e^t = (e^{\frac{1}{2}t})^2$, $t \in \mathfrak{g}$.
$\rho(e^t) = \rho(e^{\frac{1}{2}t})^2$, and the square of a unitary or antiunitary map is always unitary. $\square$

Ref$^{\underline{s}}$: • Tuynman & Wiegerinck, "Central Extensions in Physics"
• Bargmann, "Note on Wigner's Theorem" ← modern proof of Wigner's th$^{\underline{m}}$.

Observation ②:

Let $x \in \mathcal{H}$. For any $g, h \in G$, by definition,

$$gh \cdot \Pi x = g \cdot (h \cdot \Pi x).$$

To put it another way: $\boxed{gh \cdot h^{-1} \cdot g^{-1} \cdot \Pi x = \Pi x}$

I.e. $gh \cdot h^{-1} \cdot g^{-1} = id_{P\mathcal{H}}$.

So: $\rho(gh)\rho(h^{-1})\rho(g^{-1})x \in U(\mathbb{C})x$ for all $x \in \mathcal{H}$.

This says that every elt$^{\underline{\pm}}$ of $\mathcal{H}$ is an eigenvector of $\rho(gh)\rho(h^{-1})\rho(g^{-1})$.
So all of $\mathcal{H}$ is a single eigenspace of $\rho(gh)\rho(h^{-1})\rho(g^{-1})$.

Proof: Say $T: \mathcal{H} \to \mathcal{H}$ has two distinct eigenvalues, $Tx = \alpha x$ & $Ty = \beta y$.
Then $T(x+y) = \alpha x + \beta y = \alpha(x+y) + \underbrace{(\beta - \alpha)}_{\neq 0} y$. $\square$

I.e. $\rho(gh)\rho(h^{-1})\rho(g^{-1}) = \underbrace{\phi(g,h)}_{\in U(\mathbb{C})} \cdot \mathbb{1}_{\mathcal{H}}$

since $\rho \in U(\mathcal{H})$ & $\phi \in U(\mathbb{C})$

So, $$\boxed{\rho(gh) = \phi(g,h)\rho(g)\rho(h)}$$ unitary projective representation

Remark: For now this is all we can say, see the "Central Extensions in Physics" paper for more. Note that if ρ can be chosen to be smooth, the above tells us that ϕ is also smooth.

Summary: G-actions on physical system $\mathcal{H}$.

$\updownarrow$

Unitary projective G-rep.

Projective reps:

Let V be a complex vector space, $\rho: G \to \mathrm{Aut}(V)$ a projective representation.

Can we get an honest G-rep out of this? YES!

Define a "twisted multiplication" on $G \times U(\mathbb{C})$:

$$(g,\alpha)(h,\beta) := \left(gh, \frac{\alpha\beta}{\Phi(g,h)}\right)$$

Aside: Call Φ the "phase function" of ρ.

Call this G_ρ. Then

$$\boxed{\begin{aligned} \rho' : G_\rho &\longrightarrow \mathrm{Aut}(V) \\ (g,\alpha) &\longmapsto \alpha\rho(g) \end{aligned}}$$

central extension of G by $U(\mathbb{C})$

is an actual representation of G_ρ.

Now we have an exact sequence

$$1 \longrightarrow U(\mathbb{C}) \longrightarrow G_\rho \longrightarrow G \longrightarrow 1$$

$$\alpha \longmapsto (1,\alpha)$$

$$(g,\alpha) \longmapsto g$$

In general, say A is abelian, and

$$1 \to A \to G' \to G \to 1$$

is exact, and A maps into $Z(G')$.

This is called a <u>central extension</u> of G by A.

<u>Morphism of central extensions:</u>

$$\begin{tikzcd} & G_1' \arrow[dd] \arrow[dr] & & \\ 1 \arrow[r] & A \arrow[ur] \arrow[dr] & & G \arrow[r] & 1. \\ & G_2' \arrow[ur] & & \end{tikzcd}$$

Break time!

Projective representations (post-break):

Setup: • V a complex vector space
• $\rho: G \longrightarrow \mathrm{Aut}(V)$ a projective representation
• $\phi(g,h)\cdot 1_V = \rho(gh)\rho(h^{-1})\rho(g^{-1})$ the phase function

By associativity of ρ,

$$\phi(gh,k)\phi(g,h) = \phi(g,hk)\phi(h,k).$$

This means $\phi: G^2 \longrightarrow \mathbb{C}^\times$ is a 2-cocycle.

Group cohomology: look it up. Short version is

$$C^k(G,\overset{\swarrow \text{G-module}}{A}) = \text{G-mod maps } G^k \to A,$$

coboundary map $\partial: C^k \to C^{k+1}$

$$H^k(G,A) = \frac{\ker \partial^k}{\operatorname{im} \partial^{k-1}}$$

Fact: Projective reps of G are classified up to equivalence by class in $H^2(G, U(\mathbb{C}))$ of ϕ.

Example of projective representation:

Let G be a connected Lie group, and let

$$\tilde{G} = \{\text{cont. paths } \gamma: [0,1] \to G \mid \gamma(0) = 1_G\}/\text{htpy}$$

the universal cover of G. Define a multiplication in $\tilde{G}$ by

$$(\gamma\cdot\eta)(t)=\gamma(t)\eta(t).$$

With this structure, $\tilde{G}$ has the structure of a group (w.r.t. standard smooth structure from G), $\tilde{G}\xrightarrow{\pi}G$ is a group homomorphism, and we have that $\tilde{G}$ is a Lie group.

Now, consider a representation of $\tilde{G}$, $\tilde{\rho}:\tilde{G}\to \mathrm{Aut}(V)$.

For $g\in G$, define $\rho(g)$ by picking a lift and applying $\tilde{\rho}$.

Extra condition: require that $\pi(g)=1_G \Rightarrow \tilde{\rho}(g)\in U(\mathbb{C})\cdot 1_V$.

Observe $\rho(gh)\rho(h^{-1})\rho(g^{-1})$ is $\tilde{\rho}$ of some lift of 1_G, so ρ is a projective representation.

<u>Theorem (see "Central exts in physics"):</u>

If G is semisimple, every projective rep of G arises in this way!

<u>Example:</u> Time-direction preserving, parity-preserving component of Lorentz group (i.e. $SO^+(1,3)$) is semisimple, and its universal cover is $SL_2(\mathbb{C})$.

Example (special orthogonal):

$SO(\mathbb{R}) = pt$ has universal cover $SO(\mathbb{R}) = pt$.

$SO(\mathbb{R}^2)$ has universal cover $\mathbb{R}$.

Def^{n}: $\mathrm{Spin}(V)$ is the universal cover of $SO(V)$, for $\dim V \geq 3$.

So: $SO(\mathbb{R}^3)$ has universal cover $\mathrm{Spin}(\mathbb{R}^3)$. Yay!!! Right?

Remark: Not all groups in physics are semisimple! For example, $\mathbb{R}^n \times \mathbb{R}^n$ translations in position & momentum is simply connected but not semisimple — our theorem will fail, and we actually have to consider a central extension called the Heisenberg group.

Let's get a concrete presentation of $\mathrm{Spin}(V)$.

Spin Groups:

Let V be a real inner product space of dimension ≥ 3.

Every nonzero $v \in V$ is a unit in $Cl(V)$, because the inner product is positive definite ($v^2 = -\|v\|^2 \cdot 1$).

Def^{n}: • $Pin(V)$ is the subgroup of $Cl(V)^{\times}$ generated by norm-1 vectors in V.

Claim: $Spin(V)$ is the intersection of $Pin(V)$ with the even subalgebra of $Cl(V)$!

Proof:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow Spin(V) \longrightarrow SO(V) \longrightarrow 0$$

exact.

$$\pm 1 \longmapsto \pm 1$$

$$v \longmapsto (x \mapsto v x v^{-1})$$

Homotopy long exact sequence:

$$\overset{1}{\cancel{\pi_1(\mathbb{Z}/2\mathbb{Z})}} \longrightarrow \pi_1 Spin(V) \longrightarrow \overset{\mathbb{Z}/2\mathbb{Z}}{\cancel{\pi_1 SO(V)}} \xrightarrow{\text{iso}} \underbrace{\pi_0(\mathbb{Z}/2\mathbb{Z})}_{= \mathbb{Z}/2\mathbb{Z}} \longrightarrow \underbrace{\pi_0 Spin(V)}_{1 \text{ b/c connected (extra stuff)}} \longrightarrow \overset{1}{\cancel{\pi_0 SO(V)}}$$

So $\pi_1(Spin(V)) = 1$. Done! $\square$