25/06/13

Atiyah-Singer Index Theorem Seminar.

Analysis of Dirac Operators (Andrew Lee).

Oh, so much analysis!

Idea: We have sections of our Clifford bundle, and we want to quantify how "smooth" they are.

Sobolev Spaces.

We first define this for the torus $\mathbb{T}^n$.

Def^n: Let S be a Clifford bundle over $\mathbb{T}^n$, and define an inner product on $C^\infty(S)$ by (for $k \in \mathbb{Z}_{>0}$),

$$\langle f, g \rangle_{\mathbb{T}^n}^k = \sum_{|\alpha| \leq k} \langle D^\alpha f, D^\alpha g \rangle_{L^2}$$ (Distributional derivatives)

Then the k^{th} Sobolev space W^k is the completion of $C^\infty(S)$ w.r.t. the norm given by $\langle \cdot, \cdot \rangle_{\mathbb{T}^n}^k$.

Def^n: For $f \in L^2(S)$, the Fourier transform of f is

$$\hat{f}(\xi) = \int_{\mathbb{T}^n} f(x) e^{-i\xi \cdot x} dx.$$

Fact: $\widehat{D^\alpha f}(\xi) = (i\xi)^\alpha \hat{f}(\xi)$ (1D).

Def$^{\underline{n}}$: The Fourier series for f is expansion as

$$f(x) = \sum_{\nu \in \mathbb{Z}^n} a_\nu e^{i\nu \cdot x}$$

where $a_\nu = \hat{f}(\nu) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(x) e^{-i\nu \cdot x} \, dx$

Theorem (Parseval):

$$\|f\|_{L^2}^2 = (2\pi)^n \sum_{\nu \in \mathbb{Z}^n} |\hat{f}(\nu)|^2$$

The Sobolev spaces W^k have equivalent inner product

$$\langle f, g \rangle_{\mathbb{T}^n}^k = (2\pi)^n \sum_{\nu \in \mathbb{Z}^n} \hat{f}(\nu) \overline{\hat{g}(\nu)} \left(1 + \|\nu\|^2\right)^k$$

Note: If $k_1 < k_2$, then $W^{k_2} \subseteq W^{k_1}$.

Theorem (Sobolev Embedding):

If $p > \frac{n}{2}$, $W^{k+p}(\mathbb{T}^n) \hookrightarrow C^k(\mathbb{T}^n)$.

Proof:

$f \in W^{k+p}$, so $\sum_{\nu \in \mathbb{Z}^n} |\hat{f}(\nu)|^2 (1 + |\nu|^2)^k (1 + |\nu|^2)^p < \infty$.

So,

$$\left(\sum |\hat{f}(\nu)|(1+|\nu|^2)^{\frac{k}{2}}\right)^2 \leq \Big[\underbrace{\sum |\hat{f}|^2(1+|\nu|^2)^{k}(1+|\nu|^2)^{p}}_{\text{finite}}\Big]\Big(\underbrace{\sum (1+|\nu|^2)^{-p}}_{\text{integrates by hyp.}}\Big)$$

So, $\sum_\nu |\nu|^k |\hat{f}(\nu)| < \infty$

$$\sum_\nu |\widehat{D^\alpha f}(\nu)| < \infty$$

So for any $|\alpha| = k$, Fourier series converges.

reconstruct f (which started as an equiv. class) using the (smooth) Fourier coeffs to define our C^k representative

$\square$

Theorem (Rellich-Kondrachov):

If $k_1 < k_2$, then $W^{k_2} \hookrightarrow W^{k_1}$ compactly.

Proof:

Let B be the unit ball in W^{k_2}.

Claim: Given $\varepsilon > 0$, there is a subspace $Z \subseteq W^{k_2}$ of finite codim. such that if $f \in B \cap Z$, then $\|f\|_{k_1} < \varepsilon$.

Explicitly, take $\{f \in W^{k_2} \mid \hat{f}(\nu) = 0 \text{ for } |\nu| < N\}$.

Finite codim? Finitely many $|\nu|<N$.

Take $f_\nu = ce^{i\nu\cdot x}$, with $\|f\|_{k_2}\leq 1$. ← sketch for basis el$^{\text{ts}}$

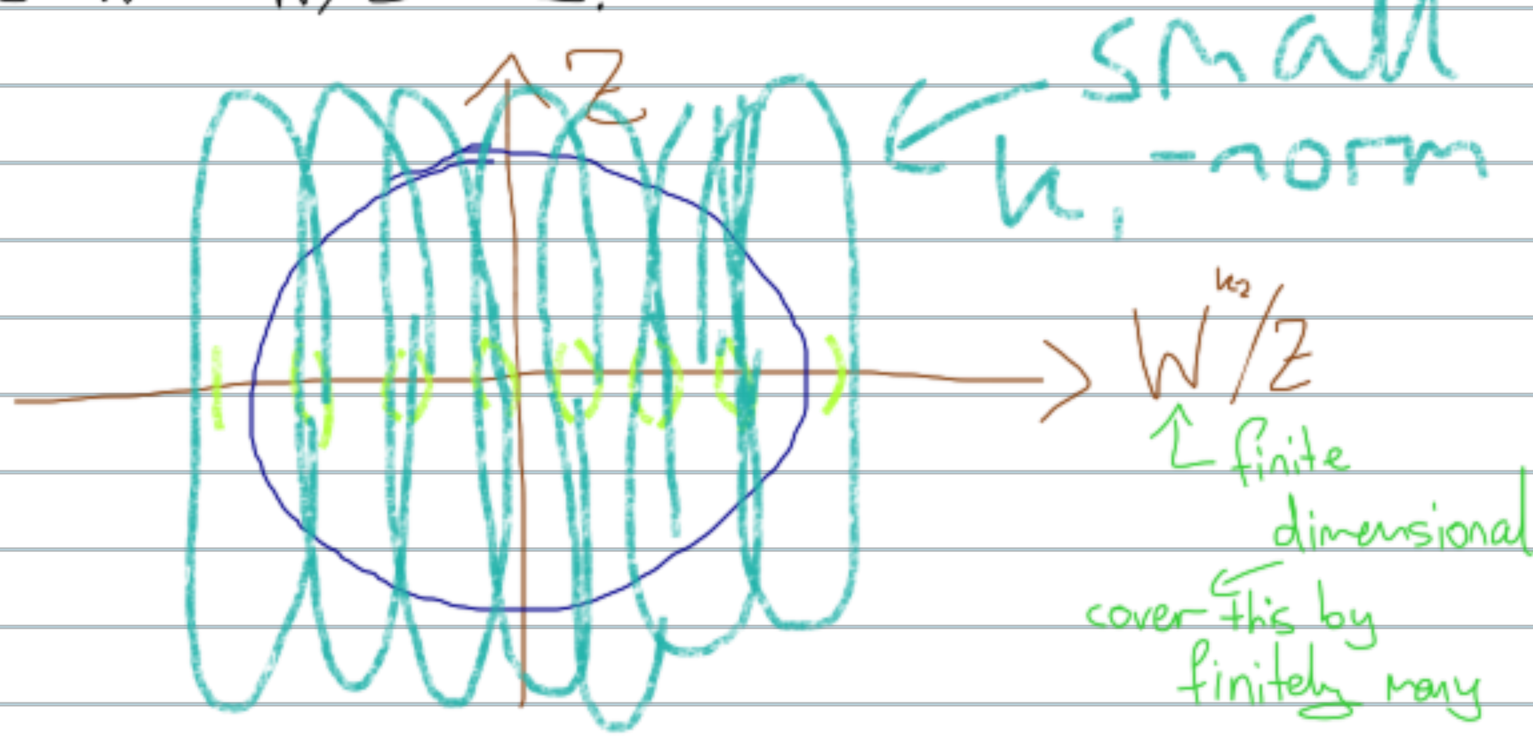
$$\|f\|_{k_2} = \int_{\mathbb{T}^n}|f|^2 + \int_{\mathbb{T}^n}|\nabla f|^2 + \cdots = c^2 + c^2\nu^2 + c^2\nu^4 + \cdots \leq 1$$

$$\|f\|_{k_1} \leq \left(\frac{1}{N^{2k_2}} + \frac{1}{N^{2k_2}}\cdot N^2 + \cdots + \frac{1}{N^{2k_2}}N^{2k_1}\right) \leq N^{\underbrace{2(k_1-k_2)}_{<0}}$$

<0, so by taking N large enough, we can make this as small as we want

(claim)

Consider $W^{k_2} = W^{k_2}/Z \oplus Z$.



Want to cover unit ball B in W^{k_2} with finitely many ε-balls.

Now, for arbitrary compact M, let $\{U_i\}$ be a cover of M, and let $\{\eta_i\}$ be a partition of unity subordinate to $\{U_i\}$, and $\{\varphi_j\}$ charts from U_j to $\mathbb{T}^n$.

Let $f, g \in L^2(S)$, $\begin{matrix} S \\ \downarrow \\ M \end{matrix}$, and define an inner product by

$$\langle f, g \rangle_M^k = \sum_j \langle (\eta_j f) \circ \varphi_j^{-1}, (\eta_j g) \circ \varphi_j^{-1} \rangle_{\mathbb{T}^n}^k .$$

Theorem (Elliptic Regularity):

For any $k > 0$, there exists a constant C_k such that for any $s \in W^k(S)$, D a Dirac operator,

$$\underbrace{\|s\|_{k+1}}_{W^{k+1} \text{ norm}} \leq C_k \left(\|s\|_k + \|Ds\|_k \right).$$

Def^n: A <u>mollifier</u> for a bundle S is a family $\{F_\varepsilon\}$, $\varepsilon \in (0,1)$, of smooth integral kernel operators on L^2 such that

1. $\{F_\varepsilon\}$ is a uniformly bounded family of operators,
2. $[B, F_\varepsilon]$ for first order B are bounded independent of ε,
3. $F_\varepsilon \to 1$ in the weak topology of operators on $L^2(S)$ i.e., for $x, y \in L^2(S)$, $\langle x, F_\varepsilon y \rangle \xrightarrow[\varepsilon \to 0]{} \langle x, y \rangle$.

$$F_\varepsilon : L^2(S) \longrightarrow L^2(S), \quad F_\varepsilon(s) = \int_{\mathbb{T}^n} \underbrace{F_\varepsilon(x,y)}_{\text{smooth}} s(x) dx$$

Theorem:

For D a Dirac operator, $\ker D$ consists of smooth sections.

Proof:

If $s \in \ker D$, suppose $s \in W^{k-1}$. Consider

$$\|F_\varepsilon s\|_k \leq C_k\left(\|F_\varepsilon s\|_{k-1} + \|D F_\varepsilon s\|_{k-1}\right)$$

$$= C_k\left(\|F_\varepsilon s\|_{k-1} + \|[D, F_\varepsilon] s\|_{k-1}\right) < \infty \quad \text{indep of } \varepsilon.$$

So F_ε are bounded in W^k, so there exists some weakly convergent subsequence in W^k

$$F_\varepsilon(s) \xrightarrow{L^2} s \quad \text{as} \quad \varepsilon \to 0.$$

So, $s \in L^2$, so $s \in W^0$, so we have by the above that $s \in W^k$ for all k, and we are done.

$\square$

Theorem (spectral-type for Dirac op.):

There is a decomposition of $L^2(S)$ into orthogonal subspaces H_λ, with each $\dim H_\lambda < \infty$ eigenspaces of D, and $\{\lambda\}$ a discrete subset of $\mathbb{R}$.

Functional Calculus/Spectral Mapping.

Need a way to obtain $F(D)$ for D a Dirac operator, f a bounded function on the spectrum of D ($\sigma(D)$).

So for $f: \sigma(D) \longrightarrow \mathbb{C}$, define a map from bounded functions on $\sigma(D)$ to bounded operators on $L^2(S)$ as

$$f \xmapsto{F} \sum_{\lambda} f(\lambda) P_\lambda \quad \leftarrow \text{proj. to } \lambda\text{-eigenspace}$$

Theorem:

The map F is a unital homomorphism from the ring of bounded functions on $\sigma(D)$ to the algebra of bounded operators on $L^2(S)$. The operator norm of $f(D)$ is bounded by $\sup_{\sigma(D)} f$.

Moreover, if A commutes with D for D Dirac and A a first order differential operator, then A commutes with $f(D)$.

If $f(x) = x g(x)$ for f, g bounded functions on $\sigma(D)$, then $f(D) = D g(D)$ as operators.

Closed Graphs.

Bounded: If $x_n \to x$, then $L(x_n) \longrightarrow L(x)$,

Closed: If $x_n \to x$ and $L(x_n)$ Cauchy, then $L(x_n) \to L(x)$.

Def$^{\underline{n}}$: The graph of an operator $L: H \to H$ is the subset of $H \times H$ given by $\{(x, Lx)\}$.

Theorem:

The closure of the graph G of a Dirac operator is also the graph of an operator $\overline{D}$.

how does this th$^{\underline{m}}$ work? properties of $\overline{D}$?

Theorem:

Suppose $x, y \in L^2(S)$, and $Dx = y$ weakly, i.e. $\langle Dx, s \rangle = \langle y, s \rangle$. Then $x \in W^1(S)$, and $\overline{D}x = y$.