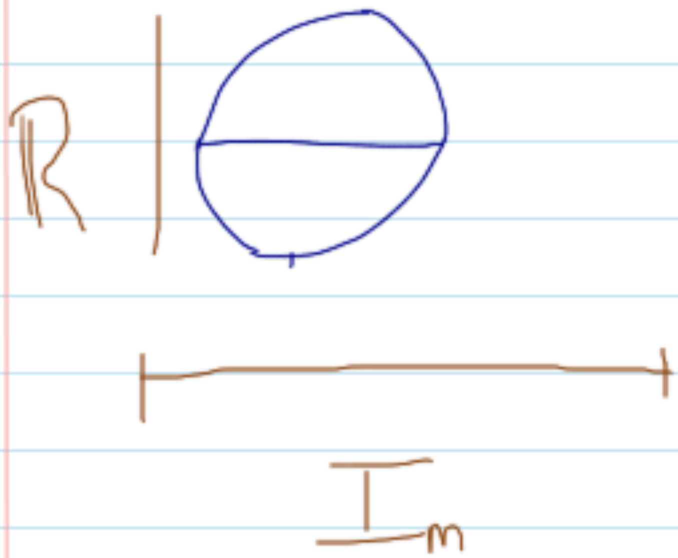
27/06/13

Atiyah-Singer Index Theorem Seminar.

Hodge Theory (Rustam Antia-Riedel & Javier Morales).

Intuition for elliptic operators & Hodge theory.

Fix $g \in C(\partial I^m)$, $\mathcal{F} = \{ f \in C^2(I_m), f|_{\partial I_m} = g \}$.



We want $f \in \mathcal{F}$ at equilibrium:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \frac{1}{\varepsilon^{m-1}} \int_{S_{\varepsilon, m_0}} f(s) - f(m_0) \, dA$$

I //

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{m-1}} \frac{1}{\varepsilon^2} \int \Big(f(m_0) + \langle \nabla f_{(m_0)}, m_0 - s \rangle + \langle H_f(s-m_0), s-m_0 \rangle - f(m_0) \Big) dA$$

$$= \frac{1}{\varepsilon^{m-1}} \int_{S_{\varepsilon, m_0}} \left\langle H_f \frac{m_0 - s}{\varepsilon}, \frac{m_0 - s}{\varepsilon} \right\rangle dA = \int_{S^{m-1}_{m_0}} \langle H_f y, y \rangle dA' = Tr\big(H_f(m_0)\big) = \Delta f(m_0),$$

where $y = \frac{m_0 - s}{\varepsilon}$, $\varepsilon^{m-1} dA' = dA$. So equilibrium $\longleftrightarrow$ Harmonic.

Let (M,g) be a Riemannian mfld, $m_0 \in M$; take $\varepsilon > 0$ s.t. $\exists N \ni m_0$, diffeomorphic to S_ε by $\exp_{m_0}$.

Now let's consider "equilibrium" again.

$$(\mathrm{I}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \frac{1}{\varepsilon^{n-1}} \int_{S_{\varepsilon, m_0}} (f(s) - f(m_0)) \overset{\text{unit normal at } s}{m_s} \lrcorner \, \mathrm{Vol}_s$$

Let γ_s be the geodesic in N from s to m_0; then (Taylor),

$$f(m_0) = f(s) + \underbrace{\varepsilon \, df_s \, \dot{\gamma}_s(0)}_{\varepsilon \langle \nabla f_s, m_s \rangle} + O(\varepsilon^2)$$

Then, $(\mathrm{I}) = \frac{1}{\varepsilon^2} \frac{1}{\varepsilon^{n-1}} \int_{S_{\varepsilon, m_0}} \varepsilon \langle \nabla f_s, m_s \rangle m_s \lrcorner \mathrm{Vol} = \frac{1}{\varepsilon^n} \int_{S_{\varepsilon, m_0}} \nabla f_s \mathrm{Vol}$

$$= \frac{1}{\varepsilon^n} \int_{S_{\varepsilon, m_0} = \partial B_{\varepsilon, m_0}} \ast df = \frac{1}{\varepsilon^n} \int_{B_{\varepsilon, m_0}} d \ast df = \int_{B_{\varepsilon, m_0}} \ast d \ast df \, \mathrm{Vol} = (d + d^*)^2 f_{m_0}.$$

Hodge theorem.

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots$$

DeRham complex for (M,g) Riemanian.

$$c \in \Omega^k(M), \quad dc = 0, \quad [c] = \{c + d\alpha \mid \alpha \in \Omega^{k-1}(M)\}.$$

affine subspace

Want to choose a preferred el$^{\pm}$ in each cohom. class. Lets consider norm minimizing c, so that given $\alpha \in \Omega^{k-1}(M)$,

$$\langle c, d\alpha \rangle = 0, \text{ so } \langle d^* c, \alpha \rangle = 0, \text{ so } d^* c = 0, \text{ so } dc = 0, \text{ so}$$

$$\Delta c = (d + d^*)^2 c = 0$$ ← Heuristic, not rigour.

Theorem (Hodge):

Each cohomology class for a Dirac complex contains a unique harmonic representative.

Proof:

Let
$$\begin{tikzcd} \mathcal{H}^1 \arrow[r, "0"] \arrow[d, "i"] & \mathcal{H}^2 \arrow[r, "0"] \arrow[d, "i"] & \mathcal{H}^3 \arrow[r, "0"] \arrow[d, "i"] & \cdots \\ C^\infty(S^1) \arrow[r, "d"] & C^\infty(S^2) \arrow[r, "d"] & C^\infty(S^3) \arrow[r] & \cdots \end{tikzcd}$$

Let $P : C^\infty(S^k) \longrightarrow \mathcal{H}^k$ be orthogonal projection.

$Pi = id_{\mathcal{H}^k} \longrightarrow$ what is iP?

$$1 - iP = dH + Hd$$

Define,
$$f(\lambda) = \begin{cases} 1, & \lambda \neq 0, \\ 0, & \lambda \neq 0. \end{cases}$$

Claim that $f(\underbrace{D}_{d+d^*}) = 1 - iP$. On the level of an e-vects. Then

for $Dv = 0$ $f(D)v = 0$ and $(1-iP)v = (1-iP)v = v - iPv = v - v = 0$; similar for nonzero e-values

Define $g = \begin{cases} \lambda^{-2}, & \lambda \neq 0, \\ 0, & \lambda = 0. \end{cases}$ $g(D) = G$, $D^2 G = f = 1 - iP$.

$$(dd^* + d^*d)G = d(d^*G) + \underbrace{(d^*G)}_{\text{this is our H, chain homotopy.}}d$$

$\blacksquare$

Corollary:

Each cohomology has finite dimension

And now for some Rustam!

Outline: • Poincaré duality.
• Something else.

Corollary (Poincaré Duality):

Let M be a compact oriented m-mfld. Then the cap product

$$\cap : H^k_{dR}(M;\mathbb{C}) \otimes H^{m-k}_{dR}(M;\mathbb{C}) \longrightarrow H^m(M;\mathbb{C}) \xrightarrow[\int_M]{\simeq} \mathbb{C}$$

is nondegenerate for all k.

Proof:

Let $0 \neq [\alpha] \in H^k_{dR}$, with harmonic rep. α (so $D^2\alpha = 0$).

If α is harmonic, so is $*\alpha$, so we have

$$\int_M \alpha \wedge *\alpha = \|\alpha\|_2 \neq 0,$$ so the pairing is nondegenerate.

$\square$

Remark: For M complex, have $d = \partial + \bar\partial$; so there are 3 Laplacians floating around: Δ_d, Δ_∂, $\Delta_{\bar\partial}$.

If M is Kähler, $\frac{1}{2}\Delta_d = \Delta_\partial = \Delta_{\bar\partial}$, and we also have the "Hodge decomp." $H^k(M;\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M;\mathbb{C})$.

Recall: M oriented, compact, $\dim M = n$.

Given an oriented closed submanifold $S \hookrightarrow M$ of $\dim S = k$,

$$[S] : H^k_{dR}(M) \longrightarrow \mathbb{C}$$
$$\alpha \longmapsto \int_S \alpha \qquad \text{(functional on } H^k\text{)}.$$

By P-D, there exists $\eta_S \in H^{n-k}$ s.t.

$$\boxed{\int_S \alpha = \int_M \alpha \wedge \eta_S}$$

Def$^{\underline{n}}$: η_S is the Poincaré dual of S.

Suppose that $R, S \hookrightarrow M$ oriented submanifolds of complementary dimensions, with $R \pitchfork S$. Then $\eta_{R\cap S} = \eta_R \wedge \eta_S$.

Let $E \xrightarrow{\pi} M$ be an oriented vector bundle over M.
Let $\operatorname{rank} E = n$.

Def$^{\underline{n}}$: The differential forms with compact vertical support are

$$\Omega^\bullet_{cv}(E) = \{\omega \in \Omega^\bullet(E) \mid \text{if } K \subset M \text{ is compact then } \pi^{-1}(K) \cap \operatorname{supp}(\omega) \text{ is compact}\}.$$

Def$^{\underline{n}}$: The cohomology with compact support is the cohomology of

$$\cdots \longrightarrow \Omega^q_{cv}(E) \to \Omega^{q+1}_{cv}(E) \to \cdots$$

Def^{n}: $\pi_*: \Omega^\bullet_{cv}(E) \to \Omega^{\bullet-n}(M)$ is called integration along fibers.

Properties:
- $d\pi_* = \pi_* d$
- projection formulas; for $\tau \in \Omega^\bullet(M)$, $\omega \in \Omega^\bullet_{cv}(E)$,
  (1) $\pi_*(\pi^*\tau \wedge \omega) = \tau \wedge \pi_*\omega$
  (2) $\int_E \pi^*\tau \wedge \omega = \int_M \tau \wedge \pi_*\omega$.
- Poincaré lemma holds: $\pi_*: H^\bullet_{cv}(M \times \mathbb{R}^n) \xrightarrow{\simeq} H^{\bullet-n}(M)$.

Theorem (Thom isomorphism theorem):

$$H^\bullet_{cv}(E) \underset{T}{\overset{\pi_*}{\rightleftarrows}} H^{\bullet-n}(M) \quad \text{is an isomorphism.}$$

Proof:

Suppose first that M is covered by two opens, $M = U \cup V$.
Then $(M-V)$,

$$0 \to \Omega^\bullet_{cv}(E|_{U\cup V}) \to \Omega^\bullet_{cv}(E|_U) \oplus \Omega^\bullet_{cv}(E|_V) \to \Omega^\bullet_{cv}(E|_{U\cap V}) \to 0$$

$$\begin{tikzcd}
H^p_{cv}(E|_{U\cup V}) \arrow[r] \arrow[d, "\pi_*"] \arrow[dr, phantom, "\circlearrowleft"] & H^p_{cv}(E|_U) \oplus H^p_{cv}(E|_V) \arrow[r] \arrow[d, "\pi_*"] \arrow[dr, phantom, "\circlearrowleft"] & H^p_{cv}(E|_{U\cap V}) \arrow[r, "\delta"] \arrow[d, "\pi_*"] & H^{p+1}_{cv}(E|_{U\cup V}) \arrow[d, "\pi_*"] \\
H^{p-n}(U\cup V) \arrow[r] & \cdots & \cdots & \cdots
\end{tikzcd}$$

Take home message is use MV - see [Bott & Tu]. $\square$

So,

$$\begin{array}{ccc} T : H^{\bullet}(M) & \longrightarrow & H_{cv}^{\bullet+n}(E) \\ \rotatebox{90}{$\in$} & & \\ 1 & \longmapsto & \Phi(E) \in H^n_{cv}(E) \end{array}$$

Thom class

distinguished elt

Proposition:

Φ is the unique class in $H^n_{cv}(E)$ s.t. for all fibres $F \subset E$, $\Phi|_F$ generates $H^n_c(F) \cong \mathbb{R}$, and

$$\int_F \Phi|_F = 1.$$

Proof:

Since $\pi_* \Phi = 1$, we have this property. Conversely, suppose Φ' had this property. We want to show that $\pi_*(\pi^*(\omega) \wedge \Phi') = \omega$, i.e. it gives the Thom iso. But,

$$\pi_*(\pi^*(\omega) \wedge \Phi') = \omega \wedge \pi_* \Phi' = \omega, \text{ so } \Phi' = \Phi. \qquad \square$$

Proposition:

Suppose $\begin{array}{c} E \\ \downarrow \\ M \end{array}, \begin{array}{c} F \\ \downarrow \\ M \end{array}$ are oriented bundles. We have

$$\begin{array}{ccc} & E \oplus F & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ E & & F \end{array}$$

and claim that $\Phi(E \oplus F) = \pi_1^* \Phi(E) \wedge \pi_2^* \Phi(F)$.

Proof: Follows from explicit description of T. $\square$

Suppose $S \hookrightarrow M$ is an oriented submfld, let T be the tubular nbhd $T \cong N(S)$. ← normal bundle.

Then $H^{\bullet}(S) \xrightarrow{\mathcal{I}} H_{cv}^{\bullet+\mathrm{codim} S}(T) \xrightarrow{j_*} H^{\bullet+\mathrm{codim} S}(M)$.

↑ extend by zero

Proposition: $\eta_S = j_* \Phi$.

Proof:

Need to show: $\int_S i^*\omega = \int_M \omega \wedge j_* \Phi$, $i: S \hookrightarrow T$, $\pi: T \to S$

We know that $\omega = \pi^* i^* \omega + d\tau$ for some τ. Calculate:

$$\int_M \omega \wedge j_* \Phi = \int_T \omega \wedge \Phi = \int_T (\pi^* i^* \omega + d\tau) \wedge \Phi = \int_T \pi^* i^* \omega \wedge \Phi = \int_M i^* \omega \wedge \pi_* \Phi = \int_M i^* \omega.$$

$\square$

Proposition:

Let $R, S \hookrightarrow M$ be oriented submanifolds of complementary dim. Then if $R \pitchfork S$, $\eta_{R\cap S} = \eta_R \wedge \eta_S$.

Proof:

Since $R \pitchfork S$, comp dim, $N(R\cap S) = N(R) \oplus N(S)$.

So $\Phi(N(R\cap S)) = \Phi(N_R) \wedge \Phi(N_S)$, and

$$\eta_{R\cap S} = \eta_R \wedge \eta_S$$

$\square$

Recall: $D^2 = \nabla^* \nabla + \underset{\substack{\parallel \\ \text{Ric}}}{K}$ if looking at ext. bundle.