11/07/13

Atiyah-Singer Index Theorem Seminar.

The Heat and Wave Equations

(Valentin Zakharevich).

Heat Equation.

On $\mathbb{R}^n$, the heat equation is

$$\frac{\partial u_t}{\partial t} - \Delta u_t(x) = 0, \qquad \text{(HE)}$$

so let's think about the case on $\mathbb{R}$. Take the Fourier transform,

$$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x) e^{-ix\xi}\, dx,$$

then the (HE) becomes

$$\frac{\partial \hat{u}}{\partial t} + \xi^2 \hat{u}_t = 0,$$

with solution $\hat{u}_t(\xi) = C(\xi) e^{-t\xi^2}$

Let v_t be such that

$$v_0 = \delta_0,$$

$$\frac{\partial v}{\partial t} - \Delta v_t = 0.$$

Then for any f,

$v_t * f$ is a solution of (HE) with $v_0 * f = f$.

Now, $\widehat{\delta_0} = \frac{1}{\sqrt{2\pi}}$, so,

$$\hat{v}_t(\xi) = \frac{1}{\sqrt{2\pi}} e^{-t\xi^2}; \text{ and,}$$

$$v_t(x) = \frac{1}{\sqrt{2\pi}} \int \frac{1}{\sqrt{2\pi}} e^{-t\xi^2} e^{ix\xi} d\xi = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$ Fundamental Solution

Let's extend to Clifford Bundles...

Let M be a compact Riemannian manifold, $S \to M$ a Clifford bundle.

We have seen the motivation $D^2 \text{"="} -\Delta$.

So we define for Clifford bundles

$$\frac{\partial s}{\partial t} + D^2 s = 0 \quad \text{(H.E.)}$$

s a section of Clifford bundle

$$\frac{\partial s}{\partial t} - iDs = 0 \quad \text{(W.E.)}$$

Recall that the Classical W.E. is

$$\frac{\partial^2 s}{\partial t^2} - \Delta s = 0.$$

Theorem:

The (WE) and (HE) have unique solutions for $s_0 \in L^2(S)$ for

$$t > 0 \quad \text{(HE)},$$
$$t \in \mathbb{R} \quad \text{(WE)}.$$

The solution satisfies:
- $\|s_t\| < \|s_0\|$ for $t > 0$ (HE),
- $\|s_t\| = \|s_0\|$ (WE).

Proof:

$$\frac{\partial}{\partial t}\|s_t\|^2 = \left\langle \frac{\partial}{\partial t}s_t, s_t\right\rangle + \left\langle s_t, \frac{\partial}{\partial t}s_t\right\rangle = -\langle D^2 s_t, s_t\rangle - \langle s_t, D^2 s_t\rangle$$
$$= -2\|Ds_t\|^2$$

heat eq^{n} version here

Then existence follows by using the solution operator(s)

$$s_t = e^{-tD^2}s_0 \quad (HE),$$

$$s_t = e^{itD}s_0 \quad (WE).$$

$\square$

Since e^{-tD^2} is a smoothing operator, there exists $k_t(p,q)$ a section of $S\boxtimes S^*$ such that

$$e^{-tD^2}s_0(p) = \int_M k_t(p,q)\,s(q)\,\mathrm{vol}(q)$$

Sobolev space

and if $s_0 \in H_\ell$ then $s_t \longrightarrow s_0$ in H_ℓ.

Define $H_{-\ell}$ as dual of H_ℓ via $\langle\, ,\rangle_0$,

$$\begin{matrix}\varphi\in H_\ell \\ \psi\in C^\infty\end{matrix} \qquad \psi.\varphi = \langle\psi,\varphi\rangle.$$

If ℓ is "big enough", then the delta function lies in $H_{-\ell}$. The solution operator then takes

$$e^{-tD^2}: H_\ell \longrightarrow H_\ell,\; H_{-\ell} \longrightarrow H_{-\ell}.$$

Theorem:

The heat kernel $k_t(p,q)$ satisfies

$$(1)\left[\frac{\partial}{\partial t}+D_p^2\right]k_t(p,q)=0.$$

If q is fixed then $k_t(\cdot,q)$ is a section of $S\otimes S_q^*$, and D_q acts on it (is the Dirac operator associated with this bundle).

$$(2)\int_M k_t(p,q)s(q)\,vol(q)\longrightarrow s(p)$$ uniformally for s smooth.

Moreover, $k_t(p,q)$ is the unique such section.

Proof:

Denote $H=\frac{\partial}{\partial t}+D_p^2$. If we can show that

$$\underbrace{H\int_M k_t(p,q)\,s_0(q)\,vol(q)}_{=0}=\int_M\left(Hk_t(p,q)\right)s_0(q)\,vol(q),$$

and since the integrand in uniformly differentiable w.r.t. t, we can bring $\frac{\partial}{\partial t}$ inside the integral, and it's sufficient to show that

$$D\int_M k_t(p,q)\,s_0(q)\,vol(q)=\int_M\left(D_p k_t(p,q)\right)s_0(q)\,vol(q).$$

Locally we can write $Ds=\sum_i e_i\nabla_i s$, and we need only show that

$$\nabla_i\int_M k_t(p,q)\,s_0(q)\,vol(q)=\int_M\left(\nabla_i k_t(p,q)\right)s(q)\,vol(q).$$

Again by uniform differentiability, can bring ∇_i inside,

$$\nabla_i \int_M k_t(p,q)\, s_0(q) \mathrm{vol}(q) = \int_M \nabla_i \left(k_t(p,q)\, s_0(q)\right) \mathrm{vol}(q).$$

Now we have to actually <u>do</u> a calculation. Oy:

$$\nabla_i\left(k_t(p,q)\sigma\right) = \left(\nabla_{i,p} k_t(p,q)\right)\sigma, \qquad s(q) = \sigma \in S_q.$$

$$k_t(p,q) = v \otimes \varphi,$$ ← write k_t as sum of these guys, use linearity

$\sigma \in S_q$, $v \in \Gamma(S)$, $\varphi \in \Gamma(S_q^*)$

$$\nabla_i(v \otimes \varphi) = \nabla_i v \otimes \varphi + v \otimes \partial_i \varphi,$$

so, $\nabla_i(v \otimes \varphi)\sigma = \nabla_i v \otimes \varphi(\sigma) + v \otimes \partial_i \varphi(\sigma)$. Applying σ first, we get

$$\nabla_i\left(v \otimes \varphi(\sigma)\right) = \nabla_i\left(\varphi(\sigma) v\right) = \underbrace{\varphi(\sigma)\nabla_i v + \partial_i \varphi(\sigma) v}_{= \nabla_i(v \otimes \varphi)\sigma},$$

proving (1).

(2) is "obvious".

<u>Uniqueness</u>: Suppose k_t' is another such heat kernel, then

$$(k_t - k_t')\, s_0 = s_t, \quad \|s_t\| < \|s_0\|,$$

and $\lim_{t \to 0} s_t = 0$ ($\|s_t\| < \varepsilon$ for any ε). So $k_t = k_t'$. $\square$

Finite Propogation Speed.

Theorem:

solution to wave eq^{n}

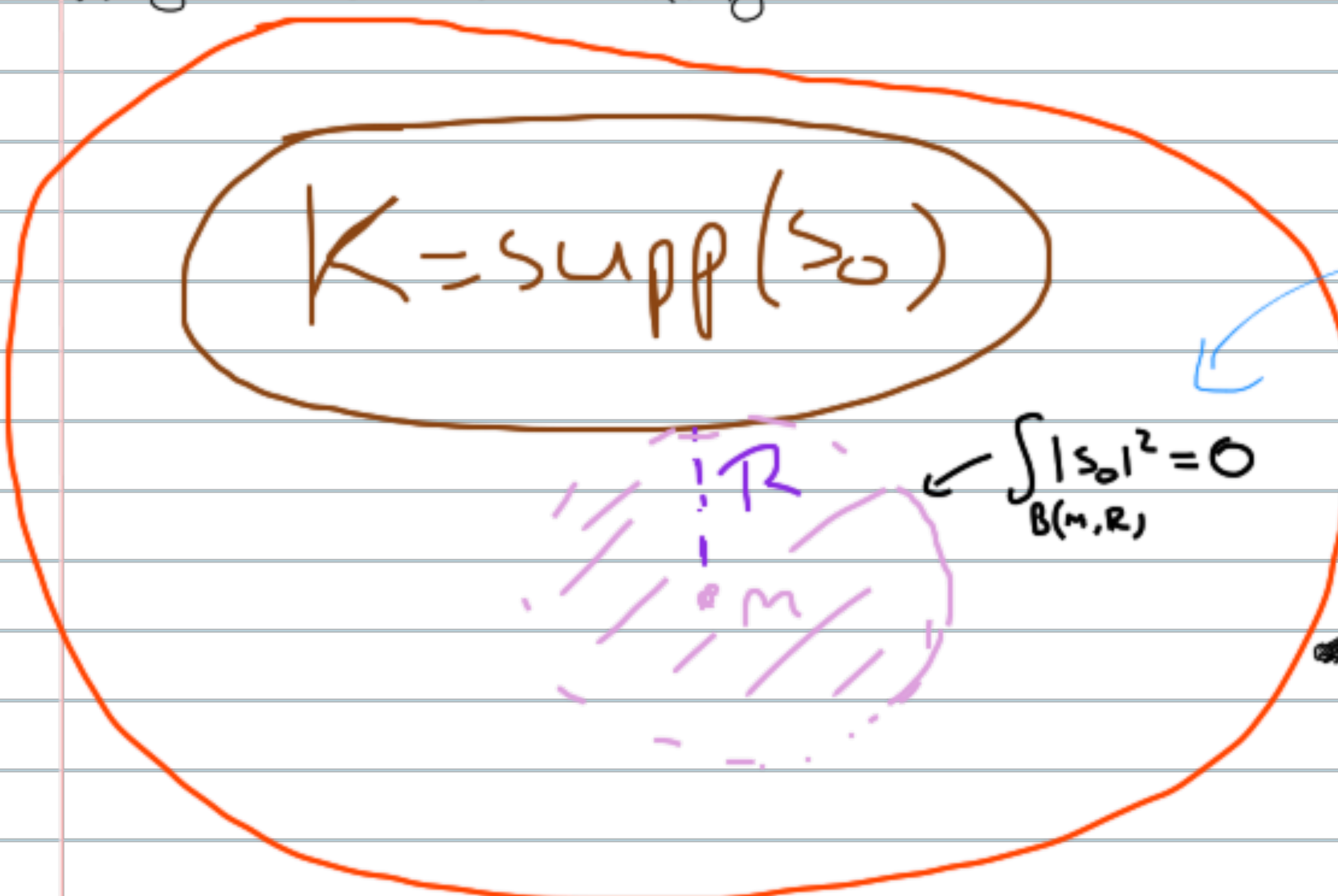
For any $s \in C_c^\infty(S)$, the support of $e^{itD}s$ lies within distance $|t|$ of $\mathrm{supp}(s)$.

Proof:

It suffices to show that

$$\int_{B(m,R-t)} |s_t|^2 \text{ is decreasing for } B(m,R) \text{ geodesic nbhd of } m.$$

Why? Stare at the diagram



So if the f^{ns} $\int_{B(m,R-t)} |s_t|^2$ is decreasing,

$\int_{B(m,R-t)} |s_t|^2 = 0$ $\forall t$.

So let's prove that the $f^{\sim}$ is decreasing. We have that

$$\langle iDs_t, s_t\rangle + \langle s_t, iDs_t\rangle = id^*\omega, \text{ where}$$

$$\omega(X) = \langle Xs_t, s_t\rangle.$$

Now,

$$\frac{\partial}{\partial t}\int_{B(m,R-t)} |s_t|^2 = \int_{S(m,R-t)} \left(-\langle s_t, s_t\rangle - \langle N\cdot s_t, s_t\rangle\right) \leq 0.$$

by Cauchy-Schwarz, somehow. $\square$

Schwarz functions

Def$^{\underline{n}}$: $S(\mathbb{R})$ = smooth functions which decay faster than any rational function

$\hookrightarrow \|X^\alpha D^\beta f\|_{L^\infty} < +\infty$.

If $f(x) = \frac{1}{\sqrt{2\pi}}\int \hat{f}(\xi) e^{ix\xi} d\xi$, can consider

$$f(D) = \frac{1}{\sqrt{2\pi}}\int \hat{f}(\xi) e^{i\xi D} d\xi.$$

Or define it in a weak sense:

$$\langle f(D)x, y\rangle = \frac{1}{\sqrt{2\pi}}\int \hat{f}(\xi)\langle e^{i\xi D}x, y\rangle d\xi.$$

Proposition:

Suppose $\mathrm{supp}(\hat{f}) \in [-c, c]$. Then $\langle f(D)x, y\rangle = 0$ for $\mathrm{dist}(\mathrm{supp}(x), \mathrm{supp}(y)) > c$.

Recall the statement from Ch5 that

$$f \mapsto \mathrm{kernel}(f(D)) \quad \text{is continuous.}$$

Our next proposition is...

Proposition:

The kernel of $f(tD)$ outside of $\Delta \subset M \times M$ $(C S \boxtimes S^*)$ tends to zero in the C^∞-topology.

Idea of proof:

We write $f_u = f_{u,1} + f_{u,2}$ where

$\hat{f}_{u,1}$ is supported on $[-\delta, \delta]$, and $\hat{f}_{u,2} \to 0$ in $S(\mathbb{R})$. //

Approximating the heat kernel for a Clifford bundle.

Def^{n}: Let $f:\mathbb{R}^+ \longrightarrow E$, E a Banach space. Then

$$f \sim \sum_{n=0}^{\infty} a_n \quad \text{(asymptotic expansion)}$$

if for all n, there exists l_n such that for all $l > l_n$,

$$\left\| f(t) - \sum_{i=0}^{l} a_i(t) \right\| < C_{l,n} |t|^n$$

for t small enough.

THE Theorem:

(i) There exists an asymptotic expansion for k_t of the form

$$k_t \sim h_t\left(\Theta_0(p,q) + t\,\Theta_1(p,q) + \cdots\right)$$

where Θ_i are smooth sections of $S \boxtimes S^*$,

$$h_t(p,q) = \frac{1}{(4\pi t)^{n/2}} e^{-d(p,q)^2/4t}.$$

(ii) The expansion is valid in $C^k(S \boxtimes S^*)$ for all $k > 0$.

(iii) Θ_i locally depends on STUFF (see Roe), and

scalar curvature at p

$$\Theta_0(p,p) = Id, \quad \Theta_1(p,p) = \frac{1}{6}\kappa(p) - K(p).$$

Clifford-contracted curvature operator appearing in Weitzenbock formula