23/07/13

Atiyah-Singer Index Theorem Seminar.

The Lefschetz Formula (Laura Starkson).

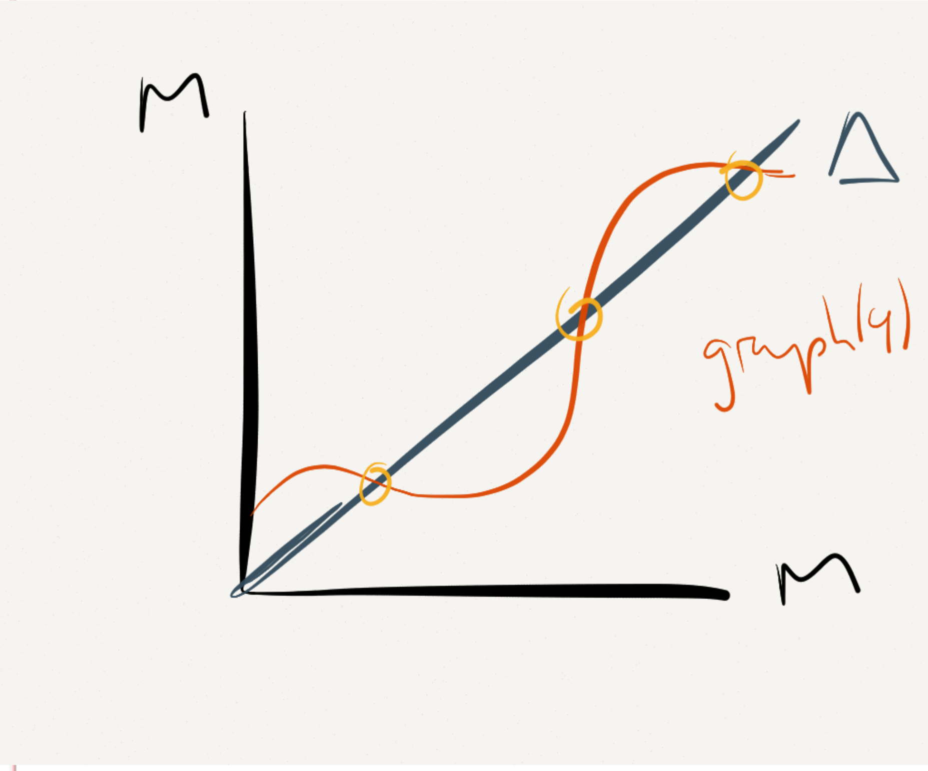
Classical Lefschetz Number.

Let M be a smooth, compact, oriented manifold, and take $\varphi: M \to M$ smooth. The Lefschetz number is a "count" of fixed points,

$$\mathrm{fix}(\varphi) = \{m \in M \mid \varphi(m) = m\}.$$

Classically, we use oriented intersection theory to define

$$L(\varphi) = I(\Delta, \mathrm{graph}(\varphi)).$$ assume $\Delta \pitchfork \mathrm{graph}(\varphi)$



It is also equivalent to $\sum_{m \in \mathrm{fix}(\varphi)} \mathrm{sgn}(m)$ where

$$\mathrm{sgn}(m) = \mathrm{sgn}(\det(I - d\varphi_m)).$$

$L(\varphi)$ is a homotopy invariant.

If $\varphi = \mathrm{id}$, $L(\varphi) = \chi(M)$.

Another formula: $L(\varphi) = \sum_{m \in \mathrm{fix}(\varphi)} \frac{\det(I - d\varphi_m)}{|\det(I - d\varphi_m)|}$

Algebraic Perspective.

We have the Thom class $\mu \in H^n(M\times M, M\times M - \Delta)$. $\leftarrow$ comes from orientation on M

The restriction $\mu' \in H^n(M\times M)$ is "dual" to Δ.

μ' is related to the fundamental class $\zeta_M \in H_n(M)$ via

$$\mu' / \zeta_M = 1 \in H^0(M)$$

$\leftarrow a\times b / \beta = a\langle b, \beta\rangle$ (slant product)

Lefschetz class of $\varphi: M \longrightarrow M$.

$$(\varphi, id): M \longrightarrow M\times M$$
$$m \longmapsto (\varphi(m), m)$$

$$\mu_\varphi := (\varphi, id)^* \mu' \in H^n(M).$$

Lefschetz number $L(\varphi) = \langle \mu_\varphi, \zeta_M \rangle$.

Lefschetz Fixed Point Theorem.

If $L(\varphi) \neq 0$ then φ has a fixed point.

From the diff. top. POV this is automatic, so let's check that the same information is being encoded in the algebraic case.

Lefschetz FPT proof (alg. def^n).

We will prove if $\mu_\varphi \neq 0$ then φ has a fixed point.

Assume (for contradiction) that φ has no fixed points.

$\mu_\varphi = (\varphi, id)^* \mu'$

$$\begin{tikzcd}
 & & M\times M - \Delta \arrow[d, hook, "i"] \\
M \arrow[r, "\Delta"] \arrow[urr, dashed, "\exists g"] \arrow[rr, bend right, "{(\varphi, id)}"'] & M\times M \arrow[r, "\varphi\times id"] & M\times M
\end{tikzcd}$$

$$m \longmapsto (m,m) \longmapsto (\varphi(m), m)$$

Now, $\mu' = j^*\mu$, $\mu \in H^n(M\times M, M\times M - \Delta)$, so

$$(\varphi, id)^* j^* \mu = \Delta^* g^* i^* j^* \mu = 0.$$

$\uparrow$ kills μ

$\square$

Theorem:

$$L(\varphi) = \sum_{q=0}^{n} (-1)^q \, tr\left(\varphi^* : H^q(M) \to H^q(M)\right).$$

Proof idea:

Write down φ^* as a matrix on H^q, pair up $\langle (\varphi, id)^* \mu', \xi_M \rangle$ $(\mu'/\xi_M = 1)$ and trace will fall out.

More detail: choose a basis $\{\alpha_i\}$ for $H^*(M)$, $\mu' \in H^n(M)$, then something to do with a Künneth number.

Take home message: It's probably just easier to do the calc. yourself. $\square$

Remark: Can define $L(\varphi)$ when M is a finite CW-complex using the formula from the theorem above.

So far, we have the following expressions for the classical Lefschetz number:

$$L(\varphi) = \sum_{m \in \mathrm{fix}(\varphi)} \frac{\det(I - d\varphi_m)}{|\det(I - d\varphi_m)|}$$

$$= \langle (\varphi, \mathrm{id})^* \mu', \zeta_M \rangle$$

$$= \sum_q (-1)^q \mathrm{tr}\left(\varphi^* : H^q(M) \longrightarrow H^q(M)\right).$$

Dirac Complex.

Recall a Dirac complex is

$$C^\infty(S_0) \xrightarrow{d} C^\infty(S_1) \xrightarrow{d} C^\infty(S_2) \xrightarrow{d} \cdots \xrightarrow{d} C^\infty(S_n)$$

where:
- $d \circ d = 0$
- $S = \bigoplus_j S_j$ is a Clifford bundle $\begin{matrix} S \\ \downarrow \\ M \end{matrix}$
- Dirac operator of S is $d + d^*$.

Examples:
- DeRham cx $S_j = \Lambda^j TM$, d
- Dolbeaux cx $S_j = \Lambda^{0,j} TM$, $\bar\partial$

Now, if we have $\varphi : M \to M$,

$$\varphi^* : C^\infty(S) \longrightarrow C^\infty(\varphi^* S)$$
$$s \longmapsto s \circ \varphi$$

fibre over m is $S_{\varphi(m)}$

Choose $\xi : \varphi^* S \to S$, and define

$$F = \xi \circ \varphi^* : C^\infty(S) \to C^\infty(S),$$

$$F(s) = \xi \circ s \circ \varphi.$$

Definition (Lefschetz number):

$$L(\xi, f) = \sum_q (-1)^q \operatorname{trace}\left(F^* : H^q(S) \to H^q(S)\right),$$

(Assuming F is a chain map, $F \circ d = d \circ F$. In this case, say (ξ, f) is a geometric endomorphism.)

If Dirac cx is

$$0 \to S^0 \xrightarrow{d} S^1 \longrightarrow 0,$$

$H^0(S) = \ker(d)$, $H^1(S) = \operatorname{coker}(d)$.

If $\varphi = id$, $\xi = id$, $id^* S = S$,

$$L(\varphi, \xi) = \dim(\ker(d)) - \dim(\operatorname{coker}(d)).$$

In general, if $\varphi = id$, $L(\varphi, \xi) = \chi(S)$.

Doing some working on the board,

$\ker d = \operatorname{coker} d^*$, $\operatorname{coker} d = \ker d^*$

$\dim(\ker(d + d^*)) - \dim(\operatorname{coker}(d + d^*))$

Something is wrong here, but we'll figure it out later...

Break time!

Consider that working from previous page:

On S^0, $d + d^* = d$.
On S^1, $d + d^* = d^*$.

So, $L(\varphi, \xi) = \dim(\ker(d + d^*) \text{ on } S^0)$
$- \dim(\ker(d + d^*) \text{ on } S^1)$

$= \dim(\ker(d)) - \dim(\ker(d^*))$

$= \dim(\ker(d)) - \dim(\operatorname{coker}(d))$. //

Goal: turn this into something from analysis.

Hodge theorem: $H^q(S) \simeq \mathcal{H}^q(S)$

harmonic sections

Write F_q^* "=" $F \circ P_q$, $P_q : L^2(S_q) \longrightarrow \mathcal{H}(S_q)$,

projection

so $tr(F_q^*) = tr(F \circ P_q)$.

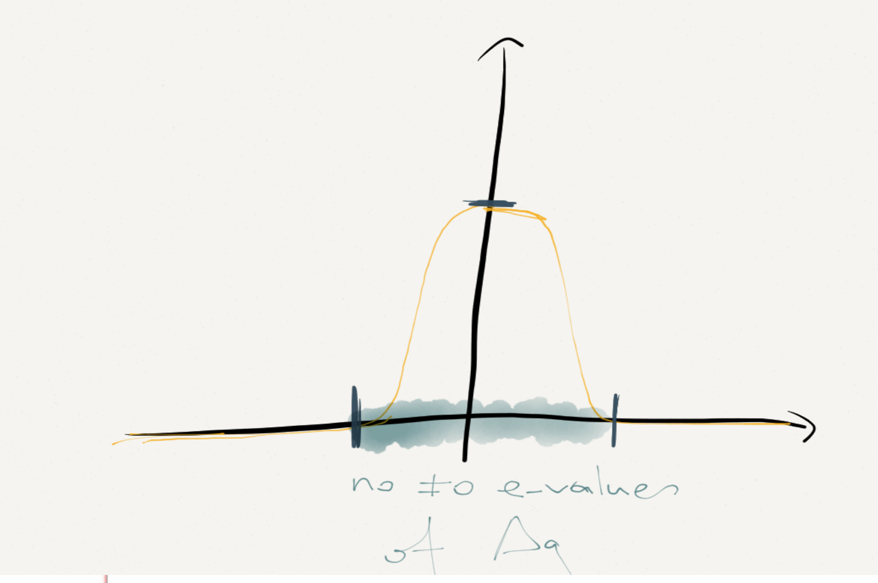
Lemma: • P_q is a smoothing operator.
• The smoothing kernel of $e^{-t\Delta_q}$ tends to the smoothing kernel of P_q as $t \to \infty$.

Proof:

Δ_q has discrete spectrum.

$$P_q = \begin{cases} 0 & \text{on } f \text{ in eigenspaces of } \Delta_q \text{ with e-value } \neq 0 \\ id & \text{on } f \text{ in the } 0 \text{ e-space of } \Delta_q \end{cases}$$

Use the functional calculus to write $P_q = f(\Delta_q)$, $\Rightarrow P_q$ is smoothing op.



$$g_t(x) = (1 - f(x))e^{-tx}$$

$=f(\Delta_q)$

$$g_t(\Delta_q) = e^{-t\Delta_q} - P_q$$

As $t \to \infty$, $g_t \to 0$, so by functional calculus, $g_t(\Delta_q) \to 0$. $\square$

So,

$$L(\xi, f) = \sum_q (-1)^q \operatorname{tr}(F \circ P_q)$$
$$= \lim_{t \to \infty} \sum_q (-1)^q \operatorname{tr}\left(F e^{-t\Delta_q}\right), \quad t > 0.$$

Claim: $\sum_q (-1)^q \operatorname{tr}\left(F e^{-t\Delta_q}\right)$ is independent of $t > 0$.

Proof:

$$\frac{d}{dt}\left(\sum_q (-1)^q \operatorname{tr}\left(F e^{-t\Delta_q}\right)\right) = \sum_q (-1)^{q+1} \operatorname{tr}\left(F(dd^* + d^*d) e^{-t\Delta_q}\right)$$

$$= \sum_q (-1)^{q+1} \left[\operatorname{tr}\left(d F d^* e^{-t\Delta_q}\right) + \operatorname{tr}\left(F d^* d e^{-t\Delta_q}\right)\right]$$

swap these (tr(AB) = tr(BA)) ← (not triv to show)

$$= \sum_q (-1)^{q+1} \left[\operatorname{tr}\left(F d^* e^{-t\Delta_q} d\right) + \operatorname{tr}\left(F d^* d e^{-t\Delta_q}\right)\right]$$

telescoping sum (w/ trivial first term, and final term killed by d(higher order) = 0)

$$= \sum_q (-1)^{q+1} \left[\operatorname{tr}\left(F d^* d e^{-t\Delta_{q-1}}\right) + \operatorname{tr}\left(F d^* d e^{-t\Delta_q}\right)\right]$$

$$= 0$$

$\square$

Lefschetz FPT: If f has no fixed points $L(\xi, f) = 0$.

Proof:

K_t^q is a section of $\begin{array}{c} S \boxtimes S^* \\ \downarrow \\ M \times M \end{array}$

Heat kernel for $e^{-t\Delta_q}$ is $K_t^q(m_1, m_2)$. ← asymptotic expansion supported near Δ as $t \to 0$

Then kernel of $Fe^{-t\Delta_q}$ is $(m_1, m_2) \mapsto \xi \cdot K_t^q(f(m_1), m_2)$.

M compact & $\mathrm{fix}(f) = \emptyset \Rightarrow$ for small enough t, $(f(m), m)$ is not in $\mathrm{supp}(K_t^q)$.

volume form

$$\text{So, } L(\xi, f) = \sum_q (-1)^q \mathrm{tr}(Fe^{-t\Delta_q}) = \sum_q (-1)^q \int_M \mathrm{tr}(\underbrace{\xi \cdot K_t^q(f(m), m)}_{=0 \text{ for small enough } t}) \, dm = 0.$$

$\square$

Atiyah-Bott fixed point theorem.

$$L(\xi, f) = \sum_{m \in \mathrm{fix}(f)} \sum_{q=0}^{n} \frac{(-1)^q \operatorname{tr}(\xi_q(m))}{|\det(1 - T_m f)|}$$

(linearization of f acting on $T_m M \to T_{f(m)} M$

Example (de Rham):

$$\xi_q : f^* \Lambda^q TM \to \Lambda^q TM$$
$$\xi_q = \Lambda^q (Tf)^*$$

Then

$$\sum_q (-1)^q \operatorname{tr}\left(\xi_q(m)\right) = \sum_q (-1)^q \left(q^{th} \text{ symm. poly}^{\text{n}} \text{ in e-values of } (T_m f)^*\right)$$
$$= \sum_q (-1)^q \det\left(I - (T_m f)^*\right)$$
$$= \sum_q (-1)^q \det\left(I - T_m f\right)$$

//

Proof will be uploaded from Laura's notes.

Remark on a lemma:

After doing some work, one obtains the approximation

$$K_t^q(f(x),x) = \frac{e^{-|x-T_0 f x|^2/4t}}{(4\pi t)^{n/2}}\left(1+O(|x|)+O(t)+O\left(\frac{|x|^3}{t}\right)\right)+O(t).$$

$$\sim \frac{e^{-\frac{S|x|^2}{4t}}}{(4\pi t)^{n/2}}\left(1+\cdots\right)$$

Upon integrating,

$$\int \frac{e^{-\frac{S|x|^2}{4t}}}{(4\pi t)^{n/2}} x^a t^b(\ldots) = O\left(t^{\frac{a}{2}+b}\right)\times(\ldots)$$