25/07/13

Atiyah-Singer Index Theorem Seminar.

The Index Problem (Valentin Zakharevich).

Index Problem.

Def^{n}: A Clifford module W is graded if $W = W_0 \oplus W_1$ and $v \in V$ interchanges the W_i.

Def^{n}: A Clifford bundle S is graded if $S = S_0 \oplus S_1$ and the interchanging action respects metric and connection.

We can think of this as a sequence

$$0 \to C^\infty(S_0) \underset{D_-}{\overset{D_+}{\rightleftarrows}} C^\infty(S_1) \to 0$$

Def^{n}: $\mathrm{Ind}(D) = \dim(\ker D_+) - \dim(\ker D_-)$.

Example: Consider the de Rham complex with grading $(-1)^q$ on Ω^q. Then $\mathrm{Ind}(D) = \chi(S)$.

Super Math!

Equivalently, a grading on a Clifford bundle is

$\varepsilon : S \to S$ a self-adjoint, parallel involution satisfying
$\varepsilon c(v) + c(v)\varepsilon = 0$.

$B(L^2(S))$ is a superalgebra with the same grading operator:

$B_0(L^2(S))$ – preserves grading
$B_1(L^2(S))$ – reverses grading.

Definition: The supertrace is $Tr_s(A) = Tr(\varepsilon A)$.

Theorem: The supertrace vanishes on supercommutators.

I.e. $Tr_s[A,B]_s = 0$ if A, B are Hilbert-Schmidt, or A is trace class.

Proof:

A, B homogeneous

$$\begin{aligned} Tr_s([A,B]_s) &= Tr\left(\varepsilon AB - \varepsilon(-1)^{d(A)d(B)} BA\right) \\ &= Tr\left((-1)^{d(A)} A \varepsilon B - (-1)^{d(A)d(B)} \varepsilon BA\right) \\ &= Tr\left(\varepsilon AB - (-1)^{(d(A)+1)d(B)} B \varepsilon A\right) \end{aligned}$$

$\square$

Prop: If A is a smoothing operator,

$$Tr_s(A) = \int_M tr_s(k(m,m))\,vol.$$

Going back to Ind(D)...

Let P be the projection onto $\ker D$. Then

$$\mathrm{Ind}(D) = \mathrm{Tr}_s(P).$$

Think: $\ker D$ is graded $(\ker D_+, \ker D_-)$, and Tr_s keeps track of this grading.

Prop: Let f be rapidly decreasing with $f(0)=1$. Then

$$\mathrm{Tr}_s(f(D^2)) = \mathrm{Ind}(D).$$

In fact it suffices that $f = O(x^{-N})$ for N large enough.

Proof:

$f = g + h$ where $g(0)=0$ and h is a bump $f^{\underline{\text{nc}}}$ containing only the zero-value of D in its support, so that $\mathrm{Ind}(D) = \mathrm{Tr}_s(h(D^2))$.

So, it suffices to show that $\mathrm{Tr}_s(g(D^2)) = 0$.

Write $g = x g_1 g_2$. Then

$$g(D^2) = D^2 g_1(D^2) g_2(D^2) \overset{\text{claim}}{=} \frac{1}{2}\left[D g_1(D^2), D g_2(D^2)\right]_s$$

Why claim? $\frac{1}{2}[\dots,\ ]_s = \frac{1}{2}\left(D^2 g_1(D^2) g_2(D^2) - (-1)^{\deg(D g_1(D^2))\deg(D g_2(D^2))} D^2 g_1(D^2) g_2(D^2)\right).$

To show: $\deg(g(D^2)) = 0$.

$$s_\lambda = s_{\lambda,+} + s_{\lambda,-}$$
$$\underset{S_0}{\cap} \qquad \underset{S_1}{\cap}$$

Apply $D^2 s_\lambda = \lambda^2 s_{\lambda,+} + \lambda^2 s_{\lambda,-}$, thus $s_{\lambda,+}$ lies in the span of the $\pm\lambda$ e-spaces.

So, $g(D^2) s_{\lambda,+} = g(\lambda^2) s_{\lambda,+} \in S_0$. $\square$

Special case: choose $f(x) = e^{-tx}$. I.e.,

$$\mathrm{Ind}(D) = \mathrm{Tr}_s(e^{-tD^2}) \sim \frac{1}{(4\pi t)^{n/2}}\left(\int_M \mathrm{tr}_s(\Theta_0)\mathrm{vol} + t\int_M \mathrm{tr}_s(\Theta_1)\mathrm{vol} + \dots\right)$$

Prop: If $\dim M$ is odd, then $\mathrm{Ind}(D) = 0$. If $\dim M$ is even then

$$\mathrm{Ind}(D) = \frac{1}{(4\pi)^{\frac{n}{2}}}\int \mathrm{tr}_s(\Theta_{\frac{n}{2}})\mathrm{vol}$$

Corollary:

If $\tilde{M} \to M$ is a k-fold covering of M and $\tilde{S}, \tilde{D}$ are natural lifts, then

$$\mathrm{Ind}(\tilde{D}) = k\,\mathrm{Ind}(D).$$

Theorem:

Let $t \mapsto D_t$ be a continuous map $[0,1] \to B(W^{k+1}, W^k)$ for all k, where D_t are Dirac operators satisfying

$$\|s\|_{k+1}^2 \leq C_k \left(\|s\|_k^2 + \|D_t\|_k^2 \right)$$

uniformally in t. Then $\mathrm{Ind}(D_0) = \mathrm{Ind}(D_1)$.

Proof:

Consider operators $(D_t \pm i)^{-1} : W^k \to W^{k+1}$. ⟵ "resolvents"

Claim that these vary continuously.

$$(D_t+i)^{-1} - (D_{t'}+i)^{-1} = (D_t+i)\underbrace{(D_{t'} - D_t)}_{(D_{t'}+i)-(D_t+i)}(D_{t'}+i)^{-1}$$

So $(1+D_t^2)^{-N}$ is cts, and so since from a previous th$^{\underline{m}}$ $\mathrm{Ind}(D_t) = \mathrm{Tr}_s(1+D_t^2)^{-N}$ is a continuously varying integer, it must be constant.

$\square$

Break Time!

Recall: In the de Rham complex Ω^q was graded by $(-1)^q$.

$$\mathrm{Ind}(D) = \frac{1}{4\pi}\int \left(\mathrm{tr}\,\Theta_1^0 - \mathrm{tr}\,\Theta_1^1 + \mathrm{tr}\,\Theta_1^2\right)\mathrm{vol},$$

$$\Theta_1 = \frac{1}{6}\kappa - \mathbf{K},$$

$$K^0 = K^2 = 0 \quad (\text{because the book says?})$$

$$K^1 = \text{Ricci curvature operator.}$$

So, $\mathrm{tr}\,\Theta_1^0 = \mathrm{tr}\,\Theta_1^2 = \frac{1}{6}K,$

$$\mathrm{tr}\,\Theta_1^1 = \frac{1}{6}K \cdot \underset{\substack{\uparrow \\ \text{1-form dim}}}{2} - K = -\frac{2}{3}K.$$

Thus,

$$\mathrm{Ind}(D) = \frac{1}{4\pi}\int_M K\,\mathrm{vol},$$

and since $\mathrm{Ind}(D) = \chi(S)$, we have proved the Gauss-Bonnet Theorem!

$$\mathrm{Ind}(D) = \frac{1}{4\pi}\int \mathrm{tr}_s(\Theta_{\frac{n}{2}})\,\mathrm{vol} \quad (\text{considering } \dim M = 2m)$$

Recall the volume element

$$\omega = e_1 \cdots e_{2m}, \quad \omega^2 = (-1)^m.$$

So we can define the <u>canonical grading</u> by $\varepsilon_0 = i^m \omega$.

Let ε be a grading operator. Then $\varepsilon\varepsilon_0$ is an involution, self-adjoint, and commutes with the Clifford action.
We get a decomp.

$$S = \underbrace{S'}_{+1 \text{ e-space}} \oplus \underbrace{S''}_{-1 \text{ e-space}} \quad (\text{for } \varepsilon\varepsilon_0)$$

So, on S', $\varepsilon = \varepsilon_0$ and on S'' $\varepsilon = -\varepsilon_0$.

So any Clifford bundle decomposes as the direct sum of a canonically graded bundle and an anticanonically graded bundle $(-\varepsilon_0)$.

So we can justify only looking at (anti)canonical bundles.

Let $S_x = \Delta \otimes V$ be canonically graded, and recall

$$\mathrm{End}\, S_x = C\ell(T_xM) \otimes \mathrm{End}(V).$$

Prop:

Let $a \otimes F \in \operatorname{End} S_x$. Then $\operatorname{tr}_s(a \otimes F) = \tau_s(a)\operatorname{tr}(F)$.

see book

Let $e_1, \dots, e_{2m}$ be an ON basis. For $E \subseteq \{1, \dots, 2m\}$,

$$\tilde{E} = \prod_{i \in E} e_i \in Cl(T_x M)$$

or any v.s.

Prop: Let $c = \sum c_E \tilde{E}$. Then $\tau_s(c) = (-2i)^m c_{\{1,\dots,2m\}}$.

Proof:

We have that $\tau_s(c) = \tau(i^m \omega c)$; so this is equivalent to $\tau(c) = 2^m c_\varnothing$. ← this is an exercise in Roe

↑ work it out as an exercise!

$\blacksquare$