06/08/13

Atiyah-Singer Index Theorem Seminar.

The Getzler Calculus (Richard Hughes).

Overview.

From last time,

$$\operatorname{Ind}(D) = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_M \operatorname{tr}_s \Theta_{\frac{n}{2}} .$$

We would like an explicit description of $\Theta_{\frac{n}{2}}$ in terms of geometric data. To do this, we will find a differential equation solved by a truncation of the heat kernel (to degree $\frac{n}{2}$), solve it explicitly, and then pick out the part of the solution corresponding to the top degree term.

In order to define the truncation and make sense of "picking out top degree terms" we need a new tool - the Getzler calculus.

Filtered algebras and symbols.

Def$^{\underline{n}}$: A graded algebra is an algebra provided with a direct sum decomposition

$$A = \bigoplus A^m$$

such that $A^m \cdot A^{m'} \subseteq A^{m+m'}$.

Examples:
- $\Lambda^\bullet V = \bigoplus_{i=0}^{\dim V} \Lambda^i V$.
- $\mathbb{C}[t] = \bigoplus_{n\geq 0} \mathbb{C}t^n$.

Def$^{\underline{n}}$: A filtration of an algebra A is a family of subspaces A_m, $m \in \mathbb{Z}$, with $A_m \subseteq A_{m+1}$ and such that $A_m \cdot A_{m'} \subseteq A_{m+m'}$ for all $m, m' \in \mathbb{Z}$.

An algebra provided with a filtration is called a filtered algebra.

Examples:
- The algebra $\mathcal{D}(M)$ of differential operators acting on functions on a manifold M is a filtered algebra with

$$\mathcal{D}_m(M) = \{\text{diff. ops of order} \leq m\}.$$

- $Cl(V)$ is a filtered algebra, with

$$Cl_m(V) = \operatorname{span}\{v_1 \cdots v_k \mid k \leq m,\ v_i \in V\}.$$

Remarks:

- Any graded algebra can be given the filtration $A_m = A^0 \oplus \cdots \oplus A^m$.
- Any homomorphic image of a filtered algebra is a filtered algebra.
- Thus we can generate a filtration any algebra from an assignment of degrees to members of some generating set (by factoring through some free algebra).

Def$^{\underline{n}}$: Let A be a filtered algebra and let G be a graded algebra. A <u>symbol map</u> $\sigma: A \to G$ is a family of linear maps $\sigma_m: A_m \longrightarrow G^m$, such that

(i) if $a \in A_{m-1}$, then $\sigma_m(a) = 0$; and,

homomorphism like property (ii) if $a \in A_m$ and $a' \in A_{m'}$, then $\sigma_m(a)\sigma_{m'}(a') = \sigma_{m+m'}(aa')$.

There is a universal symbol map we can assign to any filtered algebra.

Def$^{\underline{n}}$: Let A be a filtered algebra. The <u>associated graded algebra</u> $G(A)$ is the direct sum

$$G(A) = \bigoplus_m A_m / A_{m-1}$$

with the product operation induced from A and symbol given by the quotient maps $A_m \to A_m/A_{m-1}$.

Example:

Let $A = Cl(V)$. Then $G(A) = \Lambda^{\bullet} V$ and the symbol maps $\delta_{\bullet}: Cl(V) \to \Lambda^{\bullet} V$ pick out the appropriate top degree part a la

$$\delta_m\Big(\underbrace{\sum v_{i_1}\cdots v_{i_m}}_{\substack{\text{not expressable as a} \\ \text{product of } <m \text{ terms}}} + (\text{lower order terms})\Big) = \sum v_{i_1}\wedge\cdots\wedge v_{i_m}.$$

Example:

Let $A = \mathcal{D}(M)$, the algebra of differential operators on M. This is filtered by the order of diff. ops.

Let V be a finite dimensional vector space, and let $\mathcal{C}(V)$ denote the algebra of constant coefficient diff. ops. acting on functions of V. Then $\mathcal{C}(V)$ is a graded algebra, with

$$\mathcal{C}^m(V) = \{\text{homogeneous diff. ops. of order } m\}.$$

Form the bundle $\mathcal{C}(TM)$ whose fibre at p is $\mathcal{C}(T_pM)$. Then the space of smooth sections

$$C^\infty(\mathcal{C}(TM)) = \Gamma(M, \mathcal{C}(TM))$$

forms a graded algebra.

We will construct a symbol map

$$\sigma_\bullet : \mathcal{D}(M) \longrightarrow C^\infty(\mathcal{C}(TM)).$$

Fix $p \in M$. Given $T \in \mathcal{D}_m(M)$, choose local coords x^i with origin p and write

$$T = \sum_{|\alpha| \leq m} c_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

in terms of these local coords.

Let $\sigma_{m,p}(T) \in \mathcal{C}^m(T_pM)$ be obtained by "freezing coeffs",

$$\sigma_{m,p}(T) = \sum_{|\alpha|=m} c_\alpha(0) \frac{\partial^m}{\partial x^\alpha}.$$

Note that this vanishes on operators of order $< m$.

Claim ①: This defⁿ is coordinate independent.

Consider T in two different coordinate systems,

$$T = \sum_{|\alpha| \leq m} c_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \sum_{|\beta| \leq m} d_\beta(y) \frac{\partial^{|\beta|}}{\partial y^\beta},$$

or explicitly,

$$T = \sum_{k=0}^{m} \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} c_{(k), i_1 \leq \cdots \leq i_k}(x) \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_k}},$$

$$T = \sum_{l=0}^{m} \sum_{1 \leq j_1 \leq \cdots \leq j_l \leq n} d_{(l), j_1 \leq \cdots \leq j_l}(y) \frac{\partial}{\partial y^{j_1}} \cdots \frac{\partial}{\partial y^{j_l}}.$$

Assume

$$\begin{tikzcd} & M \arrow[dl, "x"'] \arrow[dr, "y"] & \\ \mathbb{R}^n \arrow[rr, "\Phi"'] & & \mathbb{R}^n \end{tikzcd}$$

Now,

$$d_{(k),j_1\le\cdots\le j_k}(0)\frac{\partial}{\partial y^{j_1}}\Big|_0\cdots\frac{\partial}{\partial y^{j_k}}\Big|_0$$
$$=\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n} d_{(k),j_1\le\cdots\le j_k}(\Phi(0))\left(\frac{\partial x^{i_1}}{\partial y^{j_1}}\cdots\frac{\partial x^{i_k}}{\partial y^{j_k}}\right)\Big|_0\frac{\partial}{\partial x^{i_1}}\Big|_0\cdots\frac{\partial}{\partial x^{i_k}}\Big|_0 ,$$

so,

$$\sum_{1\le j_1\le\cdots\le j_k\le n} d_{(k),j_1\le\cdots\le j_k}(0)\frac{\partial}{\partial y^{j_1}}\Big|_0\cdots\frac{\partial}{\partial y^{j_k}}\Big|_0$$
$$=\sum_{1\le j_1\le\cdots\le j_k\le n}\ \sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n} d_{(k),j_1\le\cdots\le j_k}(\Phi(0))\left(\frac{\partial x^{i_1}}{\partial y^{j_1}}\cdots\frac{\partial x^{i_k}}{\partial y^{j_k}}\right)\Big|_0\frac{\partial}{\partial x^{i_1}}\Big|_0\cdots\frac{\partial}{\partial x^{i_k}}\Big|_0$$

same const. of proportion

$$=\sum_{1\le i_1\le\cdots\le i_k\le n}\underbrace{\left(\sum_{j_1=1}^{n}\cdots\sum_{j_k=1}^{n} d_{(k),j_1\le\cdots\le j_k}(\Phi(0))\left(\frac{\partial x^{i_1}}{\partial y^{j_1}}\cdots\frac{\partial x^{i_k}}{\partial y^{j_k}}\right)\Big|_0\right.}_{C_{(k),i_1\le\cdots\le i_k}(0)}\frac{\partial}{\partial x^{i_1}}\Big|_0\cdots\frac{\partial}{\partial x^{i_k}}\Big|_0$$

$$=\sum_{1\le i_1\le\cdots\le i_k\le n} C_{(k),i_1\le\cdots\le i_k}(0)\frac{\partial}{\partial x^{i_1}}\Big|_0\cdots\frac{\partial}{\partial x^{i_k}}\Big|_0$$

and so $\sigma_{m,p}(T)$ is well-defined.

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Claim ②: If $T \in \mathcal{D}_m(M)$ and $T' \in \mathcal{D}_{m'}(M)$, then
$$\sigma_{m+m'}(TT') = \sigma_m(T)\sigma_{m'}(T).$$

Locally, write

$$T = \sum_{|\alpha| \leq m} c_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \quad T' = \sum_{|\alpha'| \leq m'} c'_{\alpha'}(x) \frac{\partial^{|\alpha'|}}{\partial x^{\alpha'}}.$$

Then,

$$TT' = \sum_{|\alpha| \leq m} \sum_{|\alpha'| \leq m'} c_\alpha(x) \underbrace{\frac{\partial^{|\alpha|}}{\partial x^\alpha}\left(c'_{\alpha'}(x) \frac{\partial^{|\alpha'|}}{\partial x^{\alpha'}}\right)}$$

by product rule, this splits into a lower graded part $\partial_\alpha(c'_{\alpha'})\partial_{\alpha'}$ and an equally graded part $c'_{\alpha'}\partial_\alpha\partial_{\alpha'}$.

If $\alpha = (i_1, \ldots, i_m)$ and $\alpha' = (j_1, \ldots, j_{m'})$, write $(\alpha, \alpha') = (i_1, \ldots, i_m, j_1, \ldots j_{m'})$. Then

$$\begin{aligned}
\sigma_{m+m'}(TT') &= \sum_{|\alpha| = m} \sum_{|\alpha'| = m'} c_\alpha(0) c'_{\alpha'}(0) \frac{\partial^m}{\partial x^\alpha} \frac{\partial^{m'}}{\partial x^{\alpha'}} \\
&= \left(\sum_{|\alpha| = m} c_\alpha(0) \frac{\partial^m}{\partial x^\alpha}\right)\left(\sum_{|\alpha'| = m'} c'_{\alpha'}(0) \frac{\partial^{m'}}{\partial x^{\alpha'}}\right) = \sigma_m(T)\sigma_{m'}(T').
\end{aligned}$$

The maps $\sigma_{m,p}$ fit together as p varies to give a linear map $\sigma_m : \mathcal{D}_m(M) \longrightarrow C^\infty(\mathcal{C}^m(TM))$, which is the desired symbol.

Remark: $\mathcal{D}(M)$ is generated by $C^\infty(M)$ in deg 0 and $\mathfrak{X}(M)$ in deg 1; the filtration on $\mathcal{D}(M)$ is determined by these generators.

To specify the symbol map, it is therefore enough to specify the action on the generators. For the preceding example, we take

$f \in C^\infty(M)$: $\sigma_0(f)$ is f itself, thought of as the operator of multiplication by the constant $f(p)$ on T_pM for each p.

$X \in \mathfrak{X}(M)$: $\sigma_1(X)$ is X itself, thought of as the constant coeff. 1st order operator $\partial_{X(p)}$ on T_pM.

Getzler symbols.

Let M be an even dimensional Riemannian manifold, and S a Clifford bundle over M.

Def$^{\underline{n}}$: The Clifford filtration on $\mathrm{End}(S)$ is given by

$$\mathrm{End}(S) = \underbrace{Cl(TM)}_{\text{standard filtration}} \otimes \underbrace{\mathrm{End}_{Cl}(S)}_{\text{degree zero}}.$$

This turns $\mathrm{End}(S)$ into a bundle of filtered algebras.

We want to study

$$\mathcal{D}(S) = \{\text{diff. ops. acting on sections of } S\}.$$

$\mathcal{D}(S)$ is gen$^{\underline{d}}$ by:
- Clifford multiplications
- covariant derivatives
- sections of $\mathrm{End}_{Cl}(S) \longrightarrow M$

In the standard filtration on $\mathcal{D}(S)$, all elements of $\mathrm{End}(S)$ would be given degree zero. By doing so, however, we lose information about the Clifford structure.

Following Getzler, we give $\mathcal{D}(S)$ a different degree assignment.

Defⁿ: The <u>Getzler filtration</u> on $\mathcal{D}(S)$ is determined by:

(i) A Clifford module endomorphism of S has degree 0.
(ii) Clifford multiplication $c(X)$, for $X \in \mathfrak{X}(M)$ has degree 1.
(iii) Covariant differentiation ∇_X, for $X \in \mathfrak{X}(M)$, has degree 1.

We will always use the Getzler filtration when considering $\mathcal{D}(S)$ as a filtered algebra.

We want a symbol map

$$\mathcal{D}(S) \longrightarrow ?$$ ← something graded

Our range will unfortunately <u>not</u> be as simple as constant coefficient differential operators anymore!

Defⁿ: Let V be a vector space, and define $\mathcal{P}(V)$ to be the algebra of polynomial coefficient differential operators acting on functions on V.

$\mathcal{P}(V)$ is a graded algebra if we give the operator $x^\alpha \partial_\beta$ the degree $|\beta| - |\alpha|$.

Example:

Recall the Riemann curvature operator $R \in \Omega^2(\mathrm{End}(TM))$, locally

$$R(\cdot) = \sum_{i<j} R(\partial_i, \partial_j)(\cdot)\, dx^i \wedge dx^j .$$

Let $X \in \mathfrak{X}(M)$, and define the map

$$\begin{aligned} T_pM &\longrightarrow \Lambda^2(T_p^*M) \\ v &\longmapsto (R_p(X_p), v). \end{aligned}$$

Identifying T with T^* via the metric gives a map

$$(R_pX_p, \cdot) : T_pM \longrightarrow \Lambda^2(T_pM)$$

and we can consider this as a degree one polynomial function on T_pM with values in $\Lambda^2 T_pM$. Putting this together gives a function

$$(RX, \cdot) \in p(TM) \otimes \Lambda^\bullet TM .$$

Soon we will want to consider what this looks like locally.

Proposition (Getzler symbol):

There is a unique symbol map

$$\sigma_\bullet : \mathcal{D}(S) \to C^\infty(p(TM) \otimes \Lambda^\bullet(TM) \otimes \mathrm{End}_{Cl}(S))$$

which has the following effect on generators:

(i) $\sigma_0(F) = F$ for a Clifford module endomorphism F;
(ii) $\sigma_1(c(X)) = e(X)$ - exterior multiplication by X - for $X \in \mathfrak{X}(M)$;
(iii) $\sigma_1(\nabla_X) = \partial_X + \frac{1}{4}(RX, \cdot)$.
note sign difference from Roe!

Remark:

Uniqueness is automatic since we have determined where each generator is sent. Existence is trickier: the specification on generators determines a unique symbol map to

$$\overset{*}{\underset{B}{\bigotimes}} V = B \oplus (B \otimes V \otimes B) \oplus (B \otimes V \otimes B \otimes V \otimes B) \oplus \cdots$$

where $B = \mathrm{End}_{Cl}(S)$ and $V = \mathfrak{X}(M) \oplus \mathfrak{X}(M)$. We need to show that this factors through the quotient map

$$\overset{*}{\underset{B}{\bigotimes}} V \twoheadrightarrow \mathcal{D}(S),$$

which we will delay until later (possibly omitting or giving only an extremely rough sketch).

Example (symbol preserves curvature identity):

In $\mathcal{D}(S)$ we have

$$\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}=K(X,Y)=R^S(X,Y)+F^S(X,Y).$$

We want to check that the symbol map described above preserves this equality (at least at second order).

Let $\{e_i\}$ be an ON basis of T_pM with associated coordinate functions $\{x^i\}$. Let $\nabla_i=\nabla_{e_i}$. Then

$$\sigma_1(\nabla_i)=\frac{\partial}{\partial x^i}-\frac{1}{4}(Re_i,\cdot).$$

Locally,

$$(Re_i,e_j)=\sum_{k<l}\left(R(e_k,e_l)e_i,e_j\right)e_k\wedge e_l,$$

so using $x^i(e_j)=\delta^i_j$ we have

$$\begin{aligned}(Re_i,\cdot)&=\sum_j\sum_{k<l}\left(R(e_k,e_l)e_i,e_j\right)x^j(\cdot)e_k\wedge e_l\\
&=\sum_j\frac{1}{2}\left[\left(\sum_{k<l}\left(R(e_k,e_l)e_i,e_j\right)x^j(\cdot)e_k\wedge e_l\right)\right.\\
&\qquad\left.+\left(\sum_{k>l}\left(R(e_k,e_l)e_i,e_j\right)x^j(\cdot)e_k\wedge e_l\right)\right]\\
&=\frac{1}{2}\sum_{j,k,l}\left(R(e_i,e_j)e_k,e_l\right)x^j(\cdot)e_k\wedge e_l,\end{aligned}$$

using that $\left(R(e_k,e_l)e_i,e_j\right)=\left(R(e_i,e_j)e_k,e_l\right)$.

So,

$$\boxed{\sigma_1(\nabla_i) = \frac{\partial}{\partial x^i} - \frac{1}{8}\sum_{j,k,l} (R(e_i,e_j)e_k, e_l)\, x^j(\cdot)\, e_k \wedge e_l\, .}$$

Now, $\sigma_2(\nabla_{[e_i,e_j]}) = 0 = \sigma_2(F^S(e_i,e_j))$ since ∇_X is degree 1 and F^S is degree 0. Letting $\tilde{R}_{ijkl} = (R(e_i,e_j)e_k, e_l)$, we calculate

$$\sigma_1(\nabla_i)\sigma_1(\nabla_{i'}) = \partial_i\partial_{i'} + \underbrace{\frac{1}{64}\sum \tilde{R}_{ijkl}\tilde{R}_{i'j'k'l'}\, x^j x^{j'}\, e_k\wedge e_l \wedge e_{k'}\wedge e_{l'}}_{\equiv © \text{, commute in } i \,\&\, i'\text{, cancel with } [\cdot,\cdot]}$$

$$- \partial_i \cdot \left(\frac{1}{8}\sum_{j,k,l} \tilde{R}_{i'jkl}\, x^j(\cdot)\, e_k\wedge e_l\right)$$

$$- \left(\frac{1}{8}\sum_{j,k,l} \tilde{R}_{ijkl}\, x^j(\cdot)\, e_k\wedge e_l\right)\cdot \partial_{i'}$$

$$= -\frac{1}{8}\sum_{j,k,l}\left(\tilde{R}_{i'jkl}\,\partial_i x^j + \tilde{R}_{ijkl}\, x^j \partial_{i'}\right) e_k\wedge e_l + ©$$

$$= © - \frac{1}{8}\sum_{j,k,l}\left(\tilde{R}_{i'jkl}\,\partial_i x^j + \tilde{R}_{ijkl}\,\partial_{i'} x^j - \delta^j_{i'}\tilde{R}_{ijkl}\right) e_k\wedge e_l,$$

using $[\partial_i, x^j] = \delta^j_i$, i.e. $x^j\partial_{i'} = \partial_{i'}x^j - \delta^j_{i'}$. So,

$$[\sigma_1(\nabla_i), \sigma_1(\nabla_{i'})] = -\frac{1}{8}\sum_{j,k,l}\Big(\tilde{R}_{i'jkl}\,\partial_i x^j + \tilde{R}_{ijkl}\,\partial_{i'} x^j - \delta^j_{i'} R_{ijkl}$$

cancel

$$- \tilde{R}_{ijkl}\,\partial_{i'} x^j - \tilde{R}_{i'jkl}\,\partial_i x^j + \delta^j_i R_{i'jkl}\Big) e_k\wedge e_l$$

$$= -\frac{1}{8}\sum_{k,l}\left(\tilde{R}_{i'ikl} - \tilde{R}_{ii'kl}\right) e_k\wedge e_l$$

$$= \frac{1}{4}\sum_{k,l}\tilde{R}_{ii'kl}\, e_k\wedge e_l = \sigma_2(R^S(e_i,e_{i'})).$$ //

Example (symbol of Dirac operator):

Locally,

$$D = \sum_i c(e_i) \nabla_i ,$$

so,

$$
\begin{aligned}
\sigma_2(D) &= \sum_i \sigma_1(c(e_i))\, \sigma_1(\nabla_i) \\
&= \sum_i e_i \partial_i - \frac{1}{8} \sum_{i,j,k,l} \big(R(e_i, e_j) e_k, e_l\big) x^j e_i \wedge e_k \wedge e_l \\
&= d_{TM} - \frac{1}{8} \sum_{i,j,k,l} R_{lkij}\, x^j e_i \wedge e_k \wedge e_l \\
&= d_{TM} + \frac{1}{8} \sum_j x^j \sum_{i,k,l} R_{jilk}\, e_i \wedge e_k \wedge e_l \\
&= d_{TM} + \frac{1}{24} \sum_j x^j \sum_{i,k,l} \big(R_{jilk}\, e_i \wedge e_k \wedge e_l + R_{jlki}\, e_l \wedge e_i \wedge e_k + R_{jkil}\, e_k \wedge e_l \wedge e_i\big) \\
&= d_{TM} + \frac{1}{24} \sum_j x^j \sum_{i,k,l} \underbrace{\big(R_{jilk} + R_{jlki} + R_{jkil}\big)}_{=0 \ (\text{Bianchi } 1)} e_i \wedge e_k \wedge e_l
\end{aligned}
$$

exterior derivative on TM

Thus,

$$\sigma_2(D) = d_{TM} .$$

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Remark: This implies that $\sigma_4(D^2) = \sigma_2(D)^2 = d_{TM}^2 = 0$.

Proposition (symbol of D^2):

The operator D^2 has Getzler order 2. Its Getzler symbol relative to an orthonormal basis of T_pM is

$$-\sum_i \left(\frac{\partial}{\partial x^i} + \frac{1}{4}\sum_j R_{ij} x^j\right)^2 + F^S$$

where

$$R_{ij} = \sum_{k<l} R_{ijkl} e_k \wedge e_l$$

is the Riemann curvature at p, and F^S is the twisting curvature 2-form at p.

Proof:

Recall that $D^2 = \nabla^* \nabla + \frac{1}{4} K + \mathbf{F}^S$, where K is the scalar curvature, and

$$\mathbf{F}^S = \sum_{i<j} c(e_i) c(e_j) F^S(e_i, e_j).$$

So $\sigma_2(K) = 0$, and

$$\sigma_2(\mathbf{F}^S) = \sum_{i<j} F^S(e_i, e_j) e_i \wedge e_j = F^S.$$

So we need to determine $\sigma_2(\nabla^* \nabla)$. At a point $p \in M$ with a synchronous frame, we have

$$\nabla^* \nabla = -\sum_i \nabla_i^2.$$

So,

$$
\begin{aligned}
\sigma_2(\nabla^*\nabla) &= -\sum_i \sigma_1(\nabla_i)^2 \\
&= -\sum_i \Big(\partial_i - \tfrac{1}{8}\sum_{j,k,l} (R(e_i,e_j)e_k, e_l)\, x^j e_k \wedge e_l\Big)^2 \\
&= -\sum_i \Big(\partial_i - \tfrac{1}{4}\sum_j \sum_{k<l} R_{lkij}\, x^j e_k \wedge e_l\Big)^2 \\
&= -\sum_i \Big(\partial_i - \tfrac{1}{4}\sum_j \sum_{k<l} R_{ijlk}\, x^j e_k \wedge e_l\Big)^2 \\
&= -\sum_i \Big(\partial_i + \tfrac{1}{4}\sum_j \sum_{k<l} R_{ijkl}\, x^j e_k \wedge e_l\Big)^2 \\
&= -\sum_i \Big(\partial_i + \tfrac{1}{4}\sum_j R_{ij}\, x^j\Big)^2
\end{aligned}
$$

Thus,

$$\sigma_2(D^2) = -\sum_i \left(\frac{\partial}{\partial x^i} + \frac{1}{4}\sum_j R_{ij} x^j\right)^2 + F^S.$$

$\square$

Getzler symbol for smoothing operators.

Since an important class of operators, smoothing operators, are <u>not</u> differential operators, we extend our definition to incorporate them. In doing so we will prove existence of the symbol defined in the previous section.

Def$^{\underline{n}}$: For a vector space V, let

$$\mathbb{C}[[V]] = \prod_{i=0}^{\infty} \otimes^i V$$

denote the <u>ring of formal power series over V</u>.

$\mathcal{P}(V) \curvearrowright \mathbb{C}[[V]]$ naturally (via term-by-term differentiation and multiplication). Let $\deg(x^\alpha) = -|\alpha|$ for $x^\alpha \in \mathbb{C}[[V]]$. Then

$$\boxed{\mathbb{C}[[V]] \text{ is a graded } \mathcal{P}(V)\text{-module.}}$$

Let $s \in C^\infty(S \boxtimes S^*) = \Gamma(M \times M, S \boxtimes S^*)$. We wish to construct a local series expansion for s.

Fix $q \in M$ and choose geodesic local coordinates x^i with origin q. Consider the function

$$\begin{aligned} M &\longrightarrow S \otimes S_q^* \\ p &\longmapsto s(p,q) \end{aligned}$$

and let $s_q(x)$ be the local coordinate representation of the function.

By Taylor's theorem, we have an asymptotic expansion of $s_q(x)$ near zero as a Taylor series

$$s_q(x) \sim \sum_\alpha s_\alpha x^\alpha$$

where the s_α are synchronous sections of $S \otimes S_q^*$ (i.e. parallel along geodesics emanating from q). Thus, each s_α is determined by its value $s_\alpha(0) \in \mathrm{End}(S_q)$, so the Taylor series can be thought of as an element of $\mathbb{C}[[T_qM]] \otimes \mathrm{End}(S_q)$.

As q varies, we obtain a section $\Sigma(s)$ of the bundle $\mathbb{C}[[TM]] \otimes \mathrm{End}(S)$.

<u>Def$^{\underline{n}}$</u>: $\Sigma(s)$ is the <u>Taylor series</u> of s.

$\mathbb{C}[[T_qM]] \otimes \mathrm{End}(S_q)$ is filtered via the tensor product filtration

$$\left(\mathbb{C}[[T_qM]] \otimes \mathrm{End}(S_q)\right)_m = \sum_{k+l=m} \left(\mathbb{C}[[T_qM]]\right)_k \otimes \left(\mathrm{End}(S_q)\right)_l,$$

where: • $\mathbb{C}[[T_qM]]$ is filtered by its grading; and,
• $\mathrm{End}(S_q)$ has the Clifford filtration.

We can use this to put a filtration on $C^\infty(S \boxtimes S^*)$:

> $s \in C^\infty(S \boxtimes S^*)$ has degree $\leq m$ if its Taylor series $\Sigma(s)$ has degree $\leq m$ at each point.

Now, $\mathrm{End}(S) \cong Cl(TM) \otimes \mathrm{End}_{Cl}(S)$, and we have the Taylor series map

$$\Sigma : C^\infty(S \boxtimes S^*) \longrightarrow C^\infty(\mathbb{C}[[TM]] \otimes \mathrm{End}(S))$$

and the Clifford symbol $Cl(TM) \longrightarrow \Lambda^\bullet(TM)$. Composing them gives a "symbol map"

$$\sigma_\bullet : C^\infty(S \boxtimes S^*) \longrightarrow C^\infty(\mathbb{C}[[TM]] \otimes \Lambda^\bullet(TM) \otimes \mathrm{End}_{Cl}(S)).$$

Def$^{\underline{n}}$: • The degree m of s relative to the above filtration is its <u>Getzler degree</u>.
• $\sigma_m(s)$ is the <u>Getzler symbol</u> of s.
• $\sigma_m^\circ(s)$ denotes the constant term in the Taylor series $\sigma_m(s)$.

Remark: The Getzler symbol does <u>not</u> have the homomorphism-like property with regard to composition of smoothing operators.

It <u>does</u> however behave well with regard to the action $\mathcal{D}(S) \circlearrowright C^\infty(S \boxtimes S^*)$, a fact we will soon exploit to prove well-definedness of the Getzler symbol for $\mathcal{D}(S)$.

Proposition (symbol respects $\mathcal{D}(S)$-action):

Let $T \in \mathcal{D}(S)$ be one of the previously described generators; i.e. a Clifford module endomorphism F, a Clifford multiplication operator $c(X)$, or a covariant derivative ∇_X. Let $m \in \{0,1\}$ be the Getzler order of T. Then for any smoothing operator Q on $C^\infty(S)$ with Getzler order $\leq k$, the smoothing operator TQ has Getzler order $\leq m+k$, and the relation

$$\sigma_{m+k}(TQ) = \sigma_m(T)\sigma_k(Q)$$

holds between symbols.

Proof idea:

Consider the kernels of Q, take its Taylor series, and then just check that the equality holds for each of the three cases. $\blacksquare$

This allows us to finally prove the well-definedness of the Getzler symbol for $\mathcal{D}(S)$.

Corollary (existence of Getzler symbol):

The Getzler symbol is well-defined on $\mathcal{D}(S)$, and satisfies the identity

$$\sigma_{m+k}(TQ) = \sigma_M(T)\sigma_k(Q)$$

for all $T \in \mathcal{D}(S)$ of Getzler order $\leq m$, and all Q of Getzler order $\leq k$.

Proof:

Given $T \in \mathcal{D}(S)$ of Getzler order $\leq m$, let $\widetilde{T}_1, \widetilde{T}_2$ be particular representations of T in terms of the three types of previously discussed generators.

Repeated application of the previous proposition yields

$$\sigma_{M+k}(TQ) = \sigma_m(\widetilde{T}_i)\sigma_k(Q), \quad i=1,2,$$

and so

$$(\sigma_m(\widetilde{T}_1) - \sigma_M(\widetilde{T}_2))\sigma_k(Q) = 0 \quad \text{for all } Q.$$

Since $\sigma_k(Q)$ is an arbitrary formal power series, we conclude that $\sigma_M(\widetilde{T}_1) = \sigma_M(\widetilde{T}_2)$. $\square$