INTRODUCTION TO LIE BIALGEBRA QUANTIZATION.

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1. Preliminary motivations.

Key Motivational Slogan: *To quantise functions on a group as a Poisson algebra, quantise an appropriate dual.*

Consider the following classical/quantum analogies:

Structure	**Classical Mechanics**	**Quantum mechanics**
Space of states	Phase space M^1	Hilbert space H
Space of observables	$C^\infty(M,\mathbb{C})$	$\mathrm{Op}(H) = \{$*all* linear operators on $H\}$
Lie structure	Poisson bracket $\{\cdot,\cdot\}$	Matrix commutator $[\cdot,\cdot]$
Hamiltonian	$\mathcal{H}_c \in C^\infty(M,\mathbb{C})$	$\mathcal{H}_q \in \mathrm{Op}(H)$
Evolution equations	$\frac{d}{dt}f(m(t)) = \{\mathcal{H}_c, f\}(m(t))$	$\frac{dA}{dt} = [\mathcal{H}_q, A]$

We would like to find a way to pass between the classical and quantum pictures: i.e., we want to quantise classical systems, and take classical limits of quantum systems.

Naive idea: $\{\cdot,\cdot\}$ and $[\cdot,\cdot]$ are both Lie brackets, so we should try to quantise Lie algebras.

Problems: (1) Lie algebras are too rigid a structure (e.g. complex simple Lie algebras have only trivial deformations); and (2) considering only the Lie bracket fails to capture all of the classical structure.

Better: Quantize Lie bialgebras and Hopf algebras.

2. Lie bialgebras and their classical doubles.

2.1. **Lie bialgebras.** Consider a Poisson-Lie group, i.e. a Lie group G with a Poisson structure, compatible in the appropriate sense. Let $\mathfrak{g} = \mathrm{Lie}(G)$. The Poisson structure induces a natural Lie algebra structure on the dual $\mathfrak{g}^*$,

$$[(df_1)_e, (df_2)_e]_{\mathfrak{g}^*} = (d\{f_1, f_2\})_e.$$

The properties of this extra structure on $\mathfrak{g}^*$ motivate the definition of a Lie bialgebra.

Definition 1. Let $\mathfrak{g}$ be a Lie algebra. A *Lie bialgebra* structure on $\mathfrak{g}$ is a skew-symmetric linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ whose dual is a Lie bracket on $\mathfrak{g}^*$, and which is a 1-cocycle of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$; i.e.

$$\delta([X,Y]) = X \cdot \delta(Y) - Y \cdot \delta(X).$$

Given a map of Poisson-Lie groups, we would like the derivative to be a map of Lie bialgebras, so we define:

Definition 2. A homomorphism of Lie bialgebras is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that $(\varphi \otimes \varphi) \circ \delta_{\mathfrak{g}} = \delta_{\mathfrak{h}} \circ \varphi$.

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2.2. **Manin triples.** We claim that $\mathfrak{g}$ and $\mathfrak{g}^*$ play symmetric roles in a Lie bialgebra – this is not immediately transparent. To make it so, we introduce Manin triples.

Definition 3. A *Manin triple* is a triple of Lie algebras $(\mathfrak{p}, \mathfrak{p}_+, \mathfrak{p}_-)$ together with an ad-invariant inner product[2] $(\cdot,\cdot)$ such that

(i) $\mathfrak{p}_\pm$ are Lie subalgebras of $\mathfrak{p}$;
(ii) $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ as vector spaces;
(iii) $\mathfrak{p}_\pm$ are isotropic for $(\cdot,\cdot)$.

Proposition 2.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then there is a 1-1 correspondence*

$$\{\textit{Lie bialgebra structures on } \mathfrak{g}\} \leftrightarrow \{\textit{Manin triples with } \mathfrak{p}_+ = \mathfrak{g}\}.$$

Proof. Given a Lie bialgebra structure on $\mathfrak{g}$ we set $\mathfrak{p}_+ = \mathfrak{g}$, $\mathfrak{p}_- = \mathfrak{g}^*$, and define the ad-invariant isotropic inner product to be the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Conversely, given such a Manin triple the pairing induces an isomorphism $\mathfrak{p}_- \cong \mathfrak{p}_+^* = \mathfrak{g}^*$, and hence a Lie algebra structure on $\mathfrak{g}^*$. To check that this is a well-defined bijection, compute by choosing a basis and a dual basis. □

Example 1. Present $\mathfrak{sl}_2\mathbb{C}$ by generators H, $X^\pm$ and relations $[X^+, X^-] = H$, $[H, X^\pm] = \pm 2X^\pm$. Then there is a "standard bialgebra structure" given by $\delta(H) = 0$ and $\delta(X^\pm) = X^\pm \wedge H$.

Example 2. Consider the current algebra for $\mathfrak{sl}_2\mathbb{C}$, $\mathfrak{g} = \mathfrak{sl}_2[u] = \mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathbb{C}[u]$ (the Lie bracket is defined term wise). Take the ad-invariant inner product on $\mathfrak{sl}_2$ given by

$$(H,H) = 2, \quad (H, X^\pm) = 0, \quad (X^\pm, X^\pm) = 0, \quad (X^\pm, X^\mp) = 1,$$

which gives us the Casimir $t = \frac{1}{2} H \otimes H + X^+ \otimes X^- + X^- \otimes X^+$. Then a Lie bialgebra structure on $\mathfrak{g}$ is given by

$$\delta(f)(u,v) = (\mathrm{ad}_{f(u)} \otimes 1 + 1 \otimes \mathrm{ad}_{f(v)}) \left(\frac{t}{u-v} \right).$$

The corresponding Manin triple is $(\mathfrak{sl}_2((u^{-1})), \mathfrak{sl}_2[u], u^{-1}\mathfrak{sl}_2[[u^{-1}]])$ with inner product

$$\langle f, g \rangle = -\mathrm{res}_0(f(u), g(u)),$$

where we have extended the inner product on $\mathfrak{sl}_2$ to a map $\mathfrak{g} \to \mathbb{C}[u]$.

2.3. **Quasitriangular Lie bialgebras.** If $\mathfrak{g}$ is a Lie bialgebra whose cocommutator is a coboundary, i.e. $\delta(X) = X \cdot r$ for some $r \in \mathfrak{g} \otimes \mathfrak{g}$, we say that it is a *coboundary Lie bialgebra.*

Proposition 2.2. *Let $\mathfrak{g}$ be a Lie algebra. Then $r \in \mathfrak{g} \otimes \mathfrak{g}$ defines a Lie bialgebra structure on $\mathfrak{g}$ if and only if*

(i) $r_{12} + r_{21}$ *is $\mathfrak{g}$-invariant; and,*
(ii) $CYBE(r) := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$ *is $\mathfrak{g}$-invariant (Classical Yang-Baxter equation).*

The easier way to satisfy condition (ii) is if $CYBE(r) = 0$, in which case r is a *classical r-matrix.*

Definition 4. A coboundary Lie bialgebra is *quasitriangular* if $CYBE(r) = 0$. It is *triangular* if furthermore, $r \in \mathfrak{g} \wedge \mathfrak{g}$.

2.4. **The classical double.** If $(\mathfrak{g}, \delta)$ is a Lie bialgebra, so is $(\mathfrak{g}, -\delta)$, which we call the *opposite* Lie bialgebra of $\mathfrak{g}$, $\mathfrak{g}^{\mathrm{op}}$.

Proposition 2.3. *Let $\mathfrak{g}$ be a finite dimensional Lie bialgebra. Then there is a canonical quasitriangular Lie bialgebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ such that the inclusions $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^* \hookleftarrow (\mathfrak{g}^*)^{op}$ are homomorphisms of Lie bialgebras.*

With this Lie bialgebra structure we call $\mathfrak{g} \oplus \mathfrak{g}^*$ the *double* of $\mathfrak{g}$ and denote it by $\mathcal{D}(\mathfrak{g})$.

[2]We don't require positive definiteness for the inner product, just nondegeneracy.

Proof. The structure is given by the element $r \in \mathfrak{g} \otimes \mathfrak{g}^* \subset \mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(\mathfrak{g})$ corresponding to the identity map of $\mathfrak{g}$. It's symmetrisation is proportional to the Casimir (hence $\mathfrak{g}$-invariant). To prove that $\mathrm{CYBE}(r) = 0$ and the inclusions are Lie bialgebra homomorphisms, choose a basis and dual basis and calculate. □

Example 3. Let $\mathfrak{g}$ be complex simple Lie algebra, and choose a Borel subalgebra $\mathfrak{b}$. $\mathfrak{b}$ can be given the structure of a Lie bialgebra [D, Example 3.2]. The double $\mathcal{D}(\mathfrak{b})$ is not *quite* the original algebra $\mathfrak{g}$, but it surjects onto $\mathfrak{g}$ as a Lie algebra with kernel a Lie bialgebra ideal. Thus, $\mathfrak{g}$ inherits a quasitriangular Lie bialgebra structure from the double $\mathcal{D}(\mathfrak{b})$.

3. Hopf algebras.

Given G a Lie group, consider $C^\infty(G)$ and $U\mathfrak{g} = (C^\infty(G))^*$. The Lie structure on G gives rise to extra structure on these function spaces, which we term a Hopf algebra.

Definition 5. A *Hopf algebra* is an algebra equipped with a compatible coalgebra structure (coassociative comultiplication $\Delta : A \to A \otimes A$ and counit $\epsilon : A \to \mathbb{C}$), and an antipode $S : A \to A$ which is the inverse to id_A under convolution of functions.

This is really representation theoretic data for the algebra A: the counit allows us to define the trivial representation, the comultiplication allows us to take tensor products of representations, and the antipode allows us to take duals of representations.

Example 4. For $\mathfrak{g}$ any Lie algebra $U\mathfrak{g}$ can be given a Hopf algebra structure by taking

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad S(x) = -x, \quad \epsilon(x) = 0,$$

for all $x \in \mathfrak{g}$.

If A is finite dimensional its dual A^* will also be a Hopf algebra. For infinite dimensional A we have to consider instead the *Hopf dual*

$$A^\circ = \{\alpha \in A^* \,|\, \mu^*(\alpha) \in A^* \otimes A^*\},$$

where $\mu : A \otimes A \to A$ is the multiplication map. Additionally, from now on many tensor products will need to be thought of topologically, not just algebraically; e.g. as the $\bullet$-adic or weak completions of the algebraic tensor product.

3.1. Coboundary structures on Hopf algebras.

Let τ be the transposition automorphism of $A \otimes A$, and define $\Delta^{\mathrm{op}}(x) = \tau \circ \Delta(x)$, the *opposite comultiplication.*

Definition 6. A Hopf algebra A is *almost cocommutative* if there exists an invertible element $\mathcal{R} \in A \otimes A$ such that for all $a \in A$,

$$\Delta^{\mathrm{op}}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}.$$

We say that $(A, \mathcal{R})$ is

- *coboundary* if $\mathcal{R}_{21} = \mathcal{R}^{-1}$ and $(\epsilon \otimes \epsilon)(\mathcal{R}) = 1$;
- *quasitriangular* if $(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{12}\mathcal{R}_{23}$ and $(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$;
- *triangular* if quasitriangular and $\mathcal{R}_{21} = \mathcal{R}^{-1}$.

In the quasitriangular case we call $\mathcal{R}$ the *universal $\mathcal{R}$-matrix* of A.[3]

A universal $\mathcal{R}$-matrix arises as a solution to the Quantum Yang-Baxter equation (QYBE)

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

The QYBE can be thought of as an exponentiated version of the CYBE: solutions to the CYBE appear as first order terms in solutions to the QYBE.

[3]The universal $\mathcal{R}$-matrix is unique up to multiplication by elements in $C(\Delta(A))$.

3.2. **A twisted tensor product for Hopf algebras.** Let B and C be Hopf algebras and $\mathcal{R} \in C \otimes B$ an invertible elements satisfying

$$(\Delta^C \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (\mathrm{id} \otimes S^B)(\mathcal{R}) = \mathcal{R}^{-1},$$
$$(\mathrm{id} \otimes \Delta^B)(\mathcal{R}) = \mathcal{R}_{12}\mathcal{R}_{13}, \qquad (S^C \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1}.$$

Then $B \otimes C$ with the usual algebra structure and

$$\Delta(b \otimes c) = \mathcal{R}_{23}\Delta^B_{13}(b)\Delta^C_{24}(c)\mathcal{R}^{-1}_{23},$$
$$S(b \otimes c) = \mathcal{R}^{-1}_{21}(S^B(b) \otimes S^C(c))\mathcal{R}_{21},$$
$$\epsilon(b \otimes c) = \epsilon^B(b)\epsilon^C(c),$$

is a Hopf algebra which we denote $B \otimes_{\mathcal{R}} C$.

3.3. **The quantum double.** Take A a Hopf algebra, and let $B = A^*$, $C = A_{\mathrm{op}}$ (opposite multiplication). One can show that the canonical element $\mathcal{R} \in A_{\mathrm{op}} \otimes A^*$ associated to the identity map satisfies the conditions above.

Definition 7. The *quantum double* of A is

$$\mathcal{D}(A) = (A^* \otimes_{\mathcal{R}} A_{\mathrm{op}})^*,$$

where $\mathcal{R}$ is the canonical element.

Analogously with the Lie bialgebra case, we have that $\mathcal{D}(A) \cong A \otimes A^*$ as coalgebras, and $A \hookrightarrow \mathcal{D}(A) \hookleftarrow (A^*)^{\mathrm{op}}$ are embeddings as Hopf algebras.

Remark The above is only true as written in the finite dimensional case. In the infinite dimensional case we need $\mathcal{R} \in A_{\mathrm{op}} \widehat{\otimes} A^\circ$ and we take

$$\mathcal{D}(A) = (A^\circ \widehat{\otimes}_{\mathcal{R}} A_{\mathrm{op}})^*.$$

As for the classical double, we have:

Proposition 3.1. *$\mathcal{D}(A)$ is quasitriangular with universal $\mathcal{R}$-matrix the identity map of A.*

4. Quantization of Hopf algebras.

4.1. **Deformations.** A deformation of a Hopf algebra $(A, \iota, \mu, \epsilon, \Delta, S)$ over $\mathbb{C}$[4] is a topological Hopf algebra $(A_h, \iota_h, \mu_h, \epsilon_h, \Delta_h, S_h)$ over $\mathbb{C}[[h]]$ such that

(i) $A_h \cong A[[h]]$ as a $\mathbb{C}[[h]]$-module;
(ii) $\mu_h \equiv \mu \mod h$ and $\Delta_h \equiv \Delta \mod h$.

We will generally just refer to the deformation as A_h when the other data is clear. Given two deformations of A, A_h and A'_h, we say $A_h \cong A'_h$ if there is an isomorphism $f_h : A_h \to A'_h$ of Hopf algebras over $\mathbb{C}[[h]]$ which is the identity $\mod h$.

The Hopf algebra structure on A_h gives rise to consistency conditions for μ_h and Δ_h, which can be expressed as cohomological conditions. [CP] define a Hopf algebra cohomology modelled on Hochschild cohomology[5] in which

$$H^2 = \{\text{space of infinitesimal deformations}\},$$
$$H^3 = \{\text{space of obstructions}\}.$$

[4] Although I have been referring to $\mathbb{C}$, all of the statements in this talk will over arbitrary fields, and most will hold over any commutative ring.

[5] This appears to be nearly the same as the "Gernstenhaber-Schack complex" constructed in [SS], but from memory the cohomology winds up shifted in degree at some point and I'm not sure where. Ultimately it's not important.

In particular, [CP] use a bicomplex $C^{p,q}$ where the $q = 1$ row encodes the algebra cohomology (standard Hochschild) and the $p = 1$ column encodes the coalgebra cohomology. This can be put to work for us, e.g.:

Proposition 4.1. *Every deformation of $U\mathfrak{g}$ for a semisimple Lie algebra $\mathfrak{g}$ is isomorphic to $U\mathfrak{g}[[h]]$ as an algebra.*

Proof. For $\mathfrak{g}$ semisimple, $H^2_{\mathrm{alg}}(U\mathfrak{g}, U\mathfrak{g}) = H^2_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{g}) = 0$. For a geometric intuition, one could think of this as a reflection of the fact that semisimple Lie algebras are classified over the discrete space of Dynkin diagrams; for an actual proof consult [W, §7.8 Semisimple Lie Algebras]. □

Remark This feature can be put to work in other ways as well: in fact by studying the cohomology of semisimple Lie algebras we can determine that there is an essentially unique deformation of $U\mathfrak{g}$ that preserves (in a precise sense) the Cartan subalgebra and triangular decomposition.

4.2. **Quasitriangular QUE algebras.**

Definition 8. A *quantised universal enveloping algebra* (QUE algebra) is a Hopf algebra deformation of $U\mathfrak{g}$ (where $\mathfrak{g}$ is any Lie algebra).

We will use the notation $U_h\mathfrak{g}$ when referring to such deformations. Be aware that unless otherwise specified, this is *not* supposed to refer to any particular deformation!

We want to study the classical degenerations $\lim_{h\to 0} U_h\mathfrak{g}$ to see what structures carry over.

Definition 9. Let $U_h\mathfrak{g}$ be a QUE algebra. If there exists $\mathcal{R}_h \in U_h\mathfrak{g} \otimes U_h\mathfrak{g}$ such that $\mathcal{R}_h \equiv 1 \otimes 1 \mod h$ and as a topological Hopf algebra

- $\mathcal{R}_{h,21} = \mathcal{R}_h^{-1}$ and $(\epsilon \otimes \epsilon)(\mathcal{R}_h) = 1$, say that $U_h\mathfrak{g}$ is *coboundary*;
- $(\Delta \otimes \mathrm{id})(\mathcal{R}_h) = \mathcal{R}_{h,12}\mathcal{R}_{h,23}$ and $(\mathrm{id} \otimes \Delta)(\mathcal{R}_h) = \mathcal{R}_{h,13}\mathcal{R}_{h,12}$, say that $U_h\mathfrak{g}$ is *quasitriangular*;
- $\mathcal{R}_{h,21} = \mathcal{R}_h^{-1}$ and $U_h\mathfrak{g}$ is quasitriangular, say that $U_h\mathfrak{g}$ is *triangular.*

The structure of a Lie bialgebra gives extra an structure to the corresponding enveloping algebra, which we call a co-Poisson-Hopf algebra (this is just the structure induced by the cocommutator on the coalgebra structure). The following proposition makes precise the sense in which Hopf algebra deformations are the "correct" way to think about Lie bialgebra deformations.

Proposition 4.2. *Let $\mathfrak{g}$ be a Lie algebra and let $U_h\mathfrak{g}$ be a Hopf algebra deformation of $U\mathfrak{g}$. Define $\delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ by*

$$\delta(x) := \frac{\Delta_h(a) - \Delta_h^{op}(a)}{h} \mod h,$$

where $x \equiv a \mod h$. Then $(U\mathfrak{g}, \delta)$ is a co-Poisson-Hopf algebra (so $(\mathfrak{g}, \delta)$ is a Lie bialgebra).

Additionally, extra structure on QUE algebras carry through to their classical limits:

Proposition 4.3. *Let $(U_h\mathfrak{g}, \mathcal{R}_h)$ be a coboundary QUE algebra, and define $r \in U\mathfrak{g} \otimes U\mathfrak{g}$ by*

$$r := \frac{\mathcal{R}_h - 1 \otimes 1}{h} \mod h.$$

Then $r \in \mathfrak{g} \otimes \mathfrak{g}$, and the classical limit of $U_h\mathfrak{g}$ is the coboundary Lie bialgebra $(\mathfrak{g}, \delta)$ defined by r. Moreover, if $(U_h\mathfrak{g}, \mathcal{R}_h)$ is (quasi)-triangular, so is $(\mathfrak{g}, \delta)$.

Proof. Choose a PBW basis and compute. Specifically, express r in the PBW basis and show that all higher degree terms must have zero coefficients. □

4.3. **QUE duals and doubles.** The dual of a QUE algebra will be a *quantum formal series Hopf algebra* (QFSH algebra): a topological Hopf algebra B over $\mathbb{C}[[h]]$ such that as a topological $\mathbb{C}[[h]]$-module $B \cong \mathbb{C}[[h]]^I$ for some index set I, and $B/hB \cong \mathbb{C}[[u_1, u_2, \ldots,]]$ as a topological algebra.

In [D, §7] Drinfel'd claims that there is an equivalence between the category of QUE algebras (with classical limit a finite dimensional Lie bialgebra) and QFSH algebras (finitely generated as topological algebras): given a QUE algebra A one can construct a canonical QFSH algebra $B \subset A$ such that the h-adic completion of B is A. A rigorous proof of this correspondence can be found in [G].

Definition 10. The *QUE dual* of A is B^*.

Since B was a QFSH algebra, we have that the Hopf dual is a QUE algebra. With this appropriate notion of the dual algebra, the construction of section 3.3 can be repeated to obtain the notion of a QUE double.

Proposition 4.4. *The classical limit of the Hopf dual of $U_h\mathfrak{g}$ will correspond to $\mathfrak{g}*$ (with Lie bialgebra structure); the classical limit of the QUE double is the classical double $\mathcal{D}(\mathfrak{g})$.*

Proof. See [G]. Really the definition has been at least partly rigged to give us the first part of the proposition; the second part should follow from the fact that taking appropriate topological duals commutes with taking appropriate topological tensor products.[6] $\square$

5. Example: Standard quantisation of $\mathfrak{sl}_2$.

Recall that $\mathfrak{sl}_2$ has a triangular Lie bialgebra structure given by $r = X^+ \wedge X^-$, i.e.

$$\delta(H) = 0, \quad \delta(X^\pm) = X^\pm \wedge H.$$

Let $\mathfrak{b}^\pm = \langle H, X^\pm \rangle \subset \mathfrak{sl}_2$ (as Lie bialgebras).

Idea: Quantize $(\mathfrak{sl}_2, \delta)$ by quantizing $(\mathfrak{b}^\pm, \delta|_{\mathfrak{b}^\pm})$.

Recall from Proposition 4.2 we want

$$\delta(x) = \frac{\Delta_h(a) - \Delta_h^{\mathrm{op}}(a)}{h} \mod h \quad \text{if } x \equiv a \mod h. \tag{5.1}$$

So look at $\mathfrak{b}^+$. We wish to find a quantisation isomorphic to $U\mathfrak{b}^+[[h]]$ as an algebra. $\delta(H) = 0$, so

$$\Delta_h(H) = H \otimes 1 + 1 \otimes H$$

satisfies (5.1). $U\mathfrak{b}^+$ is graded by $\deg(H) = 0$ and $\deg(X^+) = 1$. We will look for a quantisation that preserves this grading. This imposes

$$\Delta_h(X^+) = X^+ \otimes f + g \otimes X^+$$

where $f, g \in U\mathfrak{h}[[h]]$, $\mathfrak{h} = \langle H \rangle$. Since the zeroth order term must be the classical comultiplication, $f, g \equiv 1 \mod h$. The coassociativity condition imposes that f and g must be group like ($\Delta(f) = f \otimes f$). By computing with power series, one determines that the group like elements of $U\mathfrak{h}[[h]]$ which are $1 \mod h$ are exactly the elements $e^{h\mu H}$ for $\mu \in \mathbb{C}[[h]]$. So for some $\mu, \nu \in \mathbb{C}[[h]]$ we write

$$\Delta_h(X^+) = X^+ \otimes e^{h\mu H} + e^{h\nu H} \otimes X^+.$$

By changing the basis $X^+ \mapsto e^{-h\nu H} X^+$ we can assume that

$$\Delta_h(X^+) = X^+ \otimes e^{h\mu H} + 1 \otimes X^+.$$

Then the first term in the power series expansion of $\Delta_h(X^+) - \Delta_h^{\mathrm{op}}(X^+)$ is

$$h(X^+ \otimes \mu H - \mu H \otimes X^+),$$

so to satisfy (5.1) we require $X^+ \wedge H \equiv X^+ \wedge \mu H \mod h$, i.e. $\mu \equiv 1 \mod h$. So, set $\mu = 1$. Then

$$\Delta_h(X^+) = X^+ \otimes e^{hH} + 1 \otimes X^+.$$

[6] Fair warning: I haven't explicitly checked that this is the correct approach.

Further algebraic analysis (look for consistency requirements imposed by the Hopf algebra structure) will lead us to define

$$S_h(H) = -H, \quad S_h(X^+) = -X^+ e^{-hH}, \quad \epsilon_h(H) = \epsilon_h(X^+) = 0.$$

We can play the same game with $(\mathfrak{b}_-, \delta|_{\mathfrak{b}^-})$ to obtain

$$\Delta_h(X^-) = X^- \otimes 1 + e^{-hH} \otimes X^-, \quad S_h(X^-) = e^{-hH} X^-, \quad \epsilon_h(X^-) = 0.$$

Let this define a coalgebra structure on $U\mathfrak{sl}_2[[h]]$ as a $\mathbb{C}[[h]]$-module. We want to define an algebra structure such that Δ_h is an algebra homomorphism; this will give us the desired quantisation $U_h\mathfrak{sl}_2$. A quick computation gives us the condition

$$\Delta_h[X^+, X^-] = [X^+, X^-] \otimes e^{hH} + e^{-hH} \otimes [X^+, X^-].$$

This condition is *not* satisfied by our original multiplication; instead it will hold if $[X^+, X^-]$ is any multiple of $e^{hH} - e^{-hH}$ (remember that these are group-like elements). In order to obtain the correct classical limit, we choose

$$[X^+, X^-] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}.$$

Finally, we remark that $U_h\mathfrak{sl}_2$ is a topologically quasi-triangular Hopf algebra (which we might expect from the fact that its classical limit is quasi-triangular). There is an elementary proof of this in [CP, §6.4] in which the form of the universal $\mathcal{R}$-matrix is magicked into existence. There is a more useful procedure modelled on the idea outline in Example 3, where one constructs the quantisation as a quotient of the quantum double of $U\mathfrak{b}^+$ and obtains an explicit description of the dual standard basis. The details for this are in [CP, §8.3].

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