THE ETINGOF-KAZHDAN CONSTRUCTION OF LIE BIALGEBRA DEFORMATIONS.

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1. MOTIVATION.

Recall that we are interested in quantising Lie bialgebras. There are natural questions we can then ask:

- Can every Lie algebra be quantised?
- Is there a functorial quantisation procedure?
- Is (quasi-)triangularity preserved under quantisation?

In [D2] Drinfeld formulates precise statements of the above (and many other) important questions. The paper by [EK] Etingof and Kazhdan provides answers to a number of questions in [D2]: in particular, they prove existence of a universal quantisation for Lie bialgebras, and show that the construction is functorial. Additionally, they prove classical *r*-matrices can be quantised, and give an explicit description of the category of representations for the double of a Lie bialgebra.

This talk:

- Outline an explicit construction for quantisation of a finite dimensional Lie bialgebra.
- Demonstrate that this construction is the quantum double of a QUE algebra.
- Brief remarks on applications to quantisation of *r*-matrices and the infinite dimensional case.

Next talk (Michael Wong): Generalize this construction to infinite dimensional Lie bialgebras and prove that the construction is functorial and universal.

2. DRINFELD CATEGORY AND THE FIBRE FUNCTOR.

2.1. Drinfeld Category. We begin with the following data:

- A finite dimensional Lie algebra \mathfrak{g} over k (particular interest: finite dimensional Manin triple).
- A g-invariant elements $\Omega \in \text{Sym}^2 \mathfrak{g}$ (particular interest: Casimir element).
- An associator $\Phi \in T_3[[h]]$ (T_3 a monodromy Lie algebra).

Define the Drinfeld category \mathcal{M} by

$Ob(\mathcal{M}) = \mathfrak{g} ext{-modules},$

$\operatorname{Hom}_{\mathcal{M}}(U, W) = \operatorname{Hom}_{\mathfrak{g}}(U, W)[[h]].$

From now on, we drop the subscript \mathcal{M} on Hom.

 \mathcal{M} has a braided monodical structure. Let tensor product be the standard tensor product \mathfrak{g} -modules. Recalling that T_3 is generated by elements t_{ij} , given $V_1, V_2, V_3 \in \mathcal{M}$ define the homomorphism $\theta : T_3[[h]] \to$ $\operatorname{End}(V_1 \otimes V_2 \otimes V_3)$ by $\theta(t_{ij}) = \Omega_{ij}$. The associativity morphism is then

$$\Phi_{V_1V_2V_3} := \theta(\Phi) \in \operatorname{Hom}((V_1 \otimes V_2) \otimes V_3), V_1 \otimes (V_2 \otimes V_3)).$$

The braiding $\beta_{V_1V_2}: V_1 \otimes V_2 \to V_2 \otimes V_1$ is given by

$$\beta = s \circ e^{h\Omega/2},$$

where s is the transposition map $V_1 \otimes V_2 \to V_2 \otimes V_1$.

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2.2. Fibre functor. Let \mathcal{A} be the category of topologically free k[[h]]-modules, $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ a finite dimension Manin triple with Casimir element Ω , and \mathcal{M} the Drinfeld category associated to \mathfrak{g} . Consider the functor

$$F: \mathcal{M} \to \mathcal{A}$$
$$F(V) = \operatorname{Hom}(U\mathfrak{g}, V)$$

which is naturally isomorphic to the forgetful functor. We wish to equip F with a tensor structure: a functorial isomorphism $J_{VW}: F(V) \otimes F(W) \to F(V \otimes W)$ such that

$$F(\Phi_{VWU})J_{V\otimes W,U}\circ(J_{VW}\otimes 1)=J_{V,W\otimes U}\circ(1\otimes J_{WU}),$$

and $J_{V1} = J_{1V} = 1$. In order to do so we consider a different realisation of the functor which, while convenient for us now, is necessary to generalise to the infinite dimensional case.

Let 1 be the one dimensional trivial representation and consider the Verma modules

$$M_{\pm} = \operatorname{Ind}_{\mathfrak{g}_{\pm}}^{\mathfrak{g}} \mathbb{1}$$

which are freely generated over $U(\mathfrak{g}_{\mp})$ by vectors 1_{\pm} such that $\mathfrak{g}_{\pm}1_{\pm} = 0$ (immediate consequence of PBW theorem). This leads directly to the following lemma:

Lemma 2.1. The assignment $1 \mapsto 1_+ \otimes 1_-$ extends to an isomorphism of \mathfrak{g} -modules $\phi : U\mathfrak{g} \to M_+ \otimes M_-$.

Proof. M_{\pm} are identified with $U\mathfrak{g}_{\mp}$, so we can think of ϕ as a linear map $U\mathfrak{g} \to U\mathfrak{g}_- \otimes U\mathfrak{g}_+$. This preserves the standard filtration so induces a map $\operatorname{Sym}(\mathfrak{g}) \to \operatorname{Sym}(\mathfrak{g}_-) \otimes \operatorname{Sym}(\mathfrak{g}_+)$, which is the map induced by the isomorphism $\mathfrak{g} \to \mathfrak{g}_- \oplus \mathfrak{g}_+$.

Thus we can think of F as the functor $F(V) = \text{Hom}(M_+ \otimes M_-, V)$. Now, there are unique g-module morphisms that send

$$i_{\pm}: M_{\pm} \to M_{\pm} \otimes M_{\pm}$$
$$i_{\pm}(1_{\pm}) = 1_{\pm} \otimes 1_{\pm}.$$

Suppressing associativity morphisms, we define the tensor structure J_{VW} by

$$J_{VW}(v \otimes w) = (v \otimes w) \circ (1 \otimes \beta_{23} \otimes 1) \circ (i_+ \otimes i_-).$$

We call the functor F equipped with the tensor structure J the *fibre functor*.

3. QUANTIZATION VIA THE ENDOMORPHISM ALGEBRA OF THE FIBRE FUNCTOR.

3.1. Hopf algebra structure on $\operatorname{End}(F)$. Let $H = \operatorname{End}(F)$ be the algebra of endomorphisms on the functor F. This is naturally isomorphic to $U\mathfrak{g}[[h]]$ via the map $\alpha : U\mathfrak{g}[[h]] \to H$ defined by $(\alpha(x)f)(y) = f(yx)$; we therefore aim to quantise \mathfrak{g} by finding a Hopf algebra structure for H.

Let $F^2 : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ be the bifunctor defined by $F^2(V, W) = F(V) \otimes F(W)$; then $\operatorname{End}(F^2) = H \otimes H$. Define a bialgebra structure by

$$\Delta : H \to H \otimes H, \qquad \qquad \epsilon : H \to k[[h]],$$

$$\Delta(a)_{V,W}(v \otimes w) J_{VW}^{-1} a_{V \otimes W} J_{VW}(v \otimes w), \qquad \qquad \epsilon(a) = a_1,$$

where $a \in H, v \in F(V)$ and $w \in F(W)$. An antipode $S: H \to H$ can be similarly defined.

Proposition 3.1. The algebra H equipped with Δ , ϵ , S is a topological Hopf algebra.

Let Δ_0 and S_0 be the standard comultiplication and antipode for $U\mathfrak{g}$. One can find an element $J \in U\mathfrak{g}^{\otimes 2}[[h]]$ (and an element $Q \in U\mathfrak{g}[[h]]$ derived from J) such that

(3.1)
$$\Delta(a) = J^{-1}\Delta_0(a)J \quad \text{and} \quad S(a) = Q^{-1}S_0(a)Q$$

Corollary 3.2. Introduce a new comultiplication and antipode on the topological Hopf algebra Ug[[h]] by

$$\Delta(x) = J^{-1}\Delta_0(x)J, \quad S(x) = Q^{-1}S_0(x)Q,$$

where Δ_0 , S_0 are the usual comultiplication and antipode. Then $U_h \mathfrak{g} := (U\mathfrak{g}[[h]], \Delta, S)$ is a topological Hopf algebra isomorphic to H.

We note here that with the algebraic structure induced on $U\mathfrak{g}[[h]]$ by H the usual comultiplication and antipode is *not* a Hopf algebra, but is instead a quasi-hopf algebra (with the associativity element Φ). The element J^{-1} therefore witnesses an equivalence of the quasi-Hopf algebra $(U\mathfrak{g}[[h]], \Phi)$ with the Hopf algebra $U_h\mathfrak{g}$.

Recall that a QUEA A is a quantisation of the Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{q}})$ if

- (i) A/hA is isomorphic to $U\mathfrak{g}$ as a Hopf algebra, and
- (ii) for any $x_0 \in \mathfrak{g}$ and any $x \in A$ equal to $x_0 \mod h$ we have

$$\frac{\Delta(x) - \Delta^{\mathrm{op}}(x)}{h} \equiv \delta(x_0) \mod h.$$

Theorem 3.3. The topological Hopf algebra $U_h \mathfrak{g}$ is a quantisation of the Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$.

Proof. Recall that $\delta_{\mathfrak{g}}(x) = [x \otimes 1 + 1 \otimes x, r]$ where $r \in \mathfrak{g} \otimes \mathfrak{g}$ is the canonical element associated to the identity. Define δ as in the statement of the theorem; we wish to show that $\delta(a) = \delta_{\mathfrak{g}}(a)$ for all $a \in \mathfrak{g}$.

Let $\{g_j^+\}$ be a basis of \mathfrak{g}_+ and $\{g_j^-\}$ be the dual basis of \mathfrak{g}_- ; the canonical element is then $r = \sum_j g_j^+ \otimes g_j^-$. We have the identities

$$e^{h\Omega/2} \equiv 1 + \frac{h\Omega}{2} \mod h^2;$$

 $\Phi \equiv 1 \mod h^2 \pmod{p}$ (required by the pentagon axiom);
 $J \equiv 1 + \frac{hr}{2} \mod h^2 \pmod{p}$ (from definition of J and identity for Φ).

By (3.1),

$$\Delta(a) \equiv \Delta_0(a) + \frac{h}{2} [\Delta_0(a), r] \mod h^2$$

and so

$$\Delta(a) - \Delta^{\mathrm{op}}(a) \equiv \frac{h}{2} [\Delta_0(a), r - sr] \mod h^2$$

Since $r + sr = \Omega$ is g-invariant, we add it to the second position of the bracket so that

$$\delta(a) = [\Delta_0(a), r] = \delta_{\mathfrak{g}}(a).$$

We note here that the element

$$R = (J^{\mathrm{op}})^{-1} e^{h\Omega/2} J \in U_h \mathfrak{g}^{\otimes 2}$$

defines a quasitriangular structure on $U_h \mathfrak{g}$ that quantises r in the sense that

$$R \equiv 1 + hr \mod h^2$$

4. $U_h \mathfrak{g}$ is a quantum double.

4.1. The algebras $U_h \mathfrak{g}_{\pm}$. Since the fibre functor F is represented by $M_+ \otimes M_- \in \mathcal{M}$, we have a homomorphism

$$\theta : \operatorname{End}(M_+ \otimes M_-) \to \operatorname{End}(F) = U_h \mathfrak{g}$$
$$\theta(a)v = v \circ a$$

where $v \in F(V)$, $V \in \mathcal{M}$ and $a \in \operatorname{End}(M_+ \otimes M_-)$. This is in fact an isomorphism, so we identify $\operatorname{End}(M_+ \otimes M_-) \cong U_h \mathfrak{g}$. Using this identification, we define

$$m_{-}:F(M_{+}) \to U_{h}\mathfrak{g} \qquad \qquad m_{+}:F(M_{-}) \to U_{h}\mathfrak{g}$$
$$m_{-}(x) = (x \otimes 1) \circ (1 \otimes i_{-}) \qquad \qquad m_{+}(x) = (1 \otimes x) \circ (i_{+} \otimes 1)$$

which are embeddings (since they are embeddings mod h). We define

$$U_h\mathfrak{g}_\pm := m_\pm(F_\mp).$$

Proposition 4.1. $U_h \mathfrak{g}_{\pm}$ are subalgebras in $U_h \mathfrak{g}$.

Proof. For m_{-} , a calculation gives

$$m_{-}(x) \circ m_{-}(y) = m_{-}(z)$$
 where $z = x \circ (y \otimes 1) \circ (1 \otimes i_{-}) \in F(M_{+}).$

One can see that $U_h \mathfrak{g}_{\pm}$ is a deformation of $U \mathfrak{g}_{\pm}$ via the linear isomorphism

$$\mu: U\mathfrak{g}_{\mp} \to U_h \mathfrak{g}_{\mp}$$
$$\mu(a)(1_+ \otimes 1_-) = a1_{\pm}$$

which is also an algebra homomorphism mod h^2 . Moreover we have the following proposition (which is true because it is true mod h):

Proposition 4.2. The map $U_h\mathfrak{g}_+ \otimes U_h\mathfrak{g}_- \to U_h\mathfrak{g}$ given by $a \otimes b \mapsto ab$ is a linear isomorphism.

4.2. QUE duals and quantum doubles. There is an alternate realisation of the universal *R*-matrix of $U_h\mathfrak{g}$ as an element of $U_h\mathfrak{g}_+ \otimes U_h\mathfrak{g}_-$ given by the unique element that satisfies the identity

$$R \circ \beta^{-1} \circ (i_+ \otimes i_-) = \beta_2$$

from this an alternate explicit realisation of R can be computed. Let

$$U_h \mathfrak{g}^*_+ := \operatorname{Hom}_{\mathcal{A}}(U_h \mathfrak{g}_{\pm}, k[[h]]).$$

This is not itself a QUEA. However given a QUEA A we can define a canonical QUEA from it by taking the h-adic completion of the subalgebra

$$\sum_{n\geq 0} h^{-n} (I^*)^n \subset A^* \otimes_{k[[h]]} k((h)),$$

where I^* is the maximal ideal in A^* . We denote this QUEA A^{\vee} and call it the *dual QUEA to A*. A^* can be recovered from A^{\vee} as (roughly)¹ the maximal Hopf subalgebra on which n^{th} iterated coproducts vanish mod h^n . We denote such subalgebras with a prime, so that for instance, $A^* = (A^{\vee})'$. The following result is due to Drinfeld [D1]:

Theorem 4.3. Consider the k[[h]]-module $D(A) = A \otimes (A^{\vee})^{op}$ with canonical element $R \in A \otimes A^* \subset A \otimes (A^{\vee})^{op}$. There is a unique structure of a topological Hopf algebra on D(A) such that

- (1) $A \otimes 1$, $1 \otimes (A^{\vee})^{op}$ are Hopf subalgebras in D(A);
- (2) R defines a quasitriangular structure on D(A); and,
- (3) the linear mapping $A \otimes (A^{\vee})^{op} \to D(A)$ given by $a \otimes b \mapsto ab$ is bijective.

Equipped with this structure D(A) is a quasitriangular QUEA which we call the quantum double of A.

¹The actual definition involves constructing a sequence of new maps $\delta_n : A \to A^{\otimes n}$ from n^{th} iterated coproducts; not the iterated coproducts themselves.

4.3. Realization of $U_h\mathfrak{g}$ as a quantum double. Given what we have already proved, in order to realise $U_h\mathfrak{g}$ as a quantum double, it is sufficient to prove that $U_h\mathfrak{g}_+$ and $U_h\mathfrak{g}_-^{\mathrm{op}}$ are QUE dual Hopf subalgebras of $U_h\mathfrak{g}$. We proceed as follows: define k[[h]]-linear maps

$$\rho_{+}: U_{h}\mathfrak{g}_{-}^{*} \to U_{h}\mathfrak{g}_{+}, \qquad \rho_{-}: U_{h}\mathfrak{g}_{+}^{*} \to U_{h}\mathfrak{g}_{-}$$

$$\rho_{+}(f) = (1 \otimes f)(R), \qquad \rho_{-}(f) = (f \otimes 1)(R).$$

Denote $U_{\pm} := \text{im } (\rho_{\pm})$: these are closed as k[[h]]-subalgebras. That $U_h \mathfrak{g}_{\pm}$ are closed under comultiplication and the antipode follows from the corresponding behaviour of the universal *R*-matrix, together with the following proposition:

Proposition 4.4. $U_h \mathfrak{g}_{\pm} \otimes_{k[[h]]} k((h))$ is the h-adic completion of $U_{\pm} \otimes_{k[[h]]} k((h))$.

Proof. Given an element $T \in U_h \mathfrak{g}_{\pm}$ one can explicitly construct a sequence $\{t_{x_m}\} \subset U_{\pm} \otimes k((h))$ such that $T = \sum_{m \ge 0} t_{x_m} h^m$. The construction proceeds by reducing mod h to obtain an element $x_0 \in U \mathfrak{g}_{\pm}$, finding a t_{x_0} such that $t_{x_0} = x_0 + O(h)$, then subtracting away t_{x_0} and iterating the procedure.

From this, we consider the properties of ρ_{\pm} as maps of Hopf algebras:

Proposition 4.5. ρ_+ is an embedding of topological Hopf algebras $(U_h \mathfrak{g}_-^{op})^* \to U_h \mathfrak{g}_+$ and ρ_- is an embedding of topological Hopf algebras $U_h \mathfrak{g}_-^* \to U_h \mathfrak{g}_+^{op}$.

Proof. Fairly straightforward computation using the properties of the R-matrix.

Finally, to complete the proof that our Hopf subalgebras are QUE dual, we use the follow propsition:

Proposition 4.6. $U_{\pm} = U_h \mathfrak{g}'_{\pm}$.

Proof. To show $U_h \mathfrak{g}'_{\pm} \subset U_{\pm}$, one explicitly represents elements of $U_h \mathfrak{g}'_{\pm}$ as a series of elements in U_+ using the elements t_x from the proof of Proposition 4.4. To show $U_{\pm} \subset U_h \mathfrak{g}'_{\pm}$ recall that R has zeroth order term 1, thus any element of the form $(R-1)^n$ will be divisible by h^n ; this is exactly the type of n^{th} order coproduct vanishing that appears in the definition of A'.

Prop 4.6 implies QUE duals. Unravelling definitions and equivalences, we have

$$(U_h\mathfrak{g}_+)'=U_+\cong (U_h\mathfrak{g}_-^{\mathrm{op}})^*,$$

which gives the desired result upon taking the appropriate completion \cdot^{\vee} of both sides.

We summarise our results in the following theorem:

Theorem 4.7. [EK, Theorem 4.13] Let \mathfrak{g}_+ be a finite dimensional Lie bialgebra and $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be the associated Manin triple. Then

- (i) There exist quantised universal enveloping algebras $U_h\mathfrak{g}$ and $U_h\mathfrak{g}_{\pm} \subset U_h\mathfrak{g}$ which are quantisations of the Lie bialgebras $\mathfrak{g}, \mathfrak{g}_{\pm} \subset \mathfrak{g}$ respectively;
- (ii) The multiplication map $U_h\mathfrak{g}_+\otimes U_h\mathfrak{g}_- \to U_h\mathfrak{g}$ is a linear isomorphism;
- (iii) The algebras $U_h \mathfrak{g}_+, U_h \mathfrak{g}_-^{op}$ are QUE duals;
- (iv) The factorisation $U_h \mathfrak{g} = U_h \mathfrak{g}_+ U_h \mathfrak{g}_-$ defines an isomorphism of $U_h \mathfrak{g}$ with the quantum double of $U_h \mathfrak{g}_+$;
- (v) $U_h \mathfrak{g}$ is isomorphic to $U\mathfrak{g}[[h]]$ as a topological algebra.

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5. FINAL REMARKS.

5.1. Quantization of classical *r*-matrices and quasitriangular bialgebras. Let A be an associative unital algebra. Recall that a classical *r*-matrix in $A \otimes A$ defines a Lie bialgebra \mathfrak{g} ; in [EK, §5] Etingof and Kazhdan prove that any such *r*-matrix can be quantised to a solution R of the QYBE satisfying $R = 1 + hr + O(h^2)$.

Roughly, the proof is: form the double of \mathfrak{g} , which will be quasitriangular with *r*-matrix \tilde{r} which projects (in some appropriate sense) down to *r*. Form the QUE algebra of the double as above, which will be quasitriangular with universal *R*-matrix \tilde{R} ; then the desired quantisation of *r* is the induced projection of \tilde{R} .

Furthermore, in [EK, §6] the above is used to prove that any (quasi)triangular Lie bialgebra (not necessarily finite dimensional) admits a (quasi)triangular QUEA quantisation; once we have proved functoriality of our quantisation procedure from Section 3 it is possible to prove that the two quantisations are isomorphic.

5.2. The infinite dimensional case. All of the constructions and proofs above are *almost* true in the infinite dimensional case: there is some topological subtlety, however, and it turns out that the correct objects to study are *equicontinuous* \mathfrak{g} -modules. For infinite dimensional \mathfrak{g} , $U\mathfrak{g}$ will **not** be an equicontinuous \mathfrak{g} -module, so our original definition of the fibre functor does not make sense. We instead replace our original definition with

$$F(V) := \operatorname{Hom}(M_{-}, M_{+}^{*} \hat{\otimes} V),$$

where M_{\pm} are the same Verma modules as before and $\hat{\otimes}$ is an appropriately completed tensor product. All modules involved are now equicontinuous, and this functor is still naturally isomorphic to the forgetful functor.

With the above definitions, suppose we are given a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$. End(F) will no longer be a topological bialgebra, so we will not obtain a quantisation of \mathfrak{g} ; however, the embedding m_+ (suitably twisted) will still determine an algebra embedding whose image is a Hopf algebra. This provides the desired quantisation of \mathfrak{g}_+ .

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