M328K Homework – due Tuesday, Nov 16, 2010

This homework is not due until after the exam (Nov 11) but if you work on these problems over the next few days you will become more familiar with the multiplicative functions we have studied since the last exam. And you will also become acquainted with a major feature of today's mathematical landscape!

One of the most important functions in mathematics is the Riemann Zeta Function. For real values of $s > 1$ it is defined by

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

This definition also makes sense for complex numbers $s = a + bi$ whose real part $a = Re(s)$ is greater than 1. (Just as for real numbers we say that an infinite series of complex numbers converges if the sequence of partial sums converges.) There is a process known as *analytic continuation* that allows us to extend the domain of definition of ζ to some other complex numbers as well; in particular, there is a natural way to extend the domain of ζ to include the entire right half of the complex plane (i.e. all $s = a + bi$ with $a > 0$) except $s = 1$.

One of the most famous questions in mathematics is to locate all the zeros of the zeta function, that is, to find all complex number s with $\zeta(s) = 0$: the Riemann Hypothesis asserts that if $s = a + bi$ and $0 < a < 1$ then $a = 1/2$. This statement is generally expected to be true (among other evidence, we know that this is true for for the first *ten trillion* zeros of ζ!) but no one has a proof. If you have one, you can claim a one million dollar reward!

In the exercises that follow, you are free to treat the series completely formally, without any care about convergence, rearranging of sums, etc. Before attempting your proofs with series, by all means write out the first few terms of each series involved, with "$+\ldots$" after them, just to be sure what you're trying to do. And you are free to treat s as a real number here, if you are unfamiliar with the complex numbers. After all, this is a Number Theory course, not an Analysis course! :-)

The main reason why this function is of interest to number theorists is that this function is so closely related to the primes:

1. Show that $\zeta(s) = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots\right)$ by invoking the Fundamental Theorem of Arithmetic. (Here the product ranges over all primes p.)

2.(a) Find a closed-form expression for the sum of the geometric series above. (You may recall that the series converges for positive s since $1/p^s < 1$.)

(b) Show that there exist infinitely many prime numbers. (Cool, eh?) (*Hint* If there were only finitely many, then $\zeta(1)$ would be finite.)

(c) Extra Credit: Take logarithms to obtain an estimate for the sum $\sum_p 1/p^s$; conclude in particular that $\sum 1/p$ is infinite (which again proves there are infinitely many primes). (Fun fact: it is conjectured that there are infinitely many *twin primes*, i.e. primes p such that $p+2$ is also prime; but it is known that the sum of the reciprocals of all the twin primes is finite! So there might be infinitely many of them but they're scarce.)

3. Show that $(\zeta(s))^2 = \sum_n \frac{\tau(n)}{n^s}$

4. Show that $\zeta(s-1)\,\zeta(s) = \sum_n \frac{\sigma(n)}{n^s}$

5. Show that $\zeta(s-1)/\zeta(s) = \sum_n \frac{\varphi(n)}{n^s}$ (Possible hint: You know what $\sum_{d|n} \varphi(d)$ is; so multiply both sides by $\zeta(s)$.)

6. Show that $1/\zeta(s) = \sum_n \frac{\mu(n)}{n^s}$

7. We may write $\zeta'(s)/(\zeta(s)) = \sum_n \frac{\Lambda(n)}{n^s}$ for some function Λ. How can $\Lambda(n)$ be computed if we know the prime factorization of n? (Warning: Λ is *not* a multiplicative function!)

8(a) Compute the cyclotomic polynomials $\Phi_d(X)$ for $d = 1, 2, 3, 4, 6, 12$.

(b) Give the prime factorization of 999,999,999,999. (Hint: it is divisible by one four-digit prime.)

(c) Show that for any integer d, $\Phi_d(X)$ is of degree $\varphi(d)$ and

$$\Phi_d(X) = X^{\varphi(d)} - \mu(d)X^{\varphi(d)-1} + \ldots$$