M361K – Spring 2023 – Rusin – HW1 – due Jan 20

Before we get to the real numbers themselves let's practice a few other things. If nothing else, this will help to establish notation!

Remember that if you are asked to "compute" something, you must demonstrate that your answers are correct. Here is an example:

PROBLEM: Prove that $2 + 2 = 4$.
RESPONSE: We claim that the answer is 4. Recall first that the symbols 2, 3, and 4 are respectively defined to mean $1 + 1$, $2 + 1$, and $3 + 1$. Thus $2 + 2 = 2 + (1 + 1)$. By the associative property $2 + (1 + 1) = (2 + 1) + 1 = 3 + 1 = 4$, as claimed.

1. For each positive integer $n \in N$ let $M_n = \{n\,x \mid x \in Z\}$. Compute

$$\bigcap_{n>0} M_n \qquad \text{and} \qquad \bigcup_{n>0} M_n$$

.

2. Suppose $f : R \to R$ is defined by $f(x) = x^2 + 1$, so that for example $f(1) = 2$ and $f(5) = 26$. Compute $f([1, 5])$ and $f^{-1}([1, 5])$.

3. Suppose that $f : A \to B$ and $g : B \to C$ are both injective (i.e. one-to-one). Show that $g \circ f$ is also injective.

4. Prove by induction that for every integer $n \in N$, $n^5 - n$ is a multiple of 5.

5. A *rational point in the plane* is a point $(x, y) \in R^2$ for which $x \in Q$ and $y \in Q$. Show that there are only countably many lines in the plane that pass through two rational points.

6. Convince me that you understand Cantor's proof in the following way: if $X = \{F, U, N\}$, then (a) list the elements of the powerset P of X; (b) find two different functions $f : X \to P$; (c) compute the sets A and B for each of your functions; (d) compute the image of each of your functions. You should see that B is not in the image of f.

What about A — can you find such a function f for which A is in the image of f? Can you find one for which it's not?