M365C (Rusin) HW3 – due Thursday, Sept 19 2019

1. Compute $\inf(S)$ and $\sup(S)$ where $S = \left\{ \dfrac{1}{n+1} - \dfrac{1}{m+1} \ : \ m, n \in \mathbf{N} \right\}$

ANSWER: the inf and sup are $+1$ and -1 respectively.

2. Define a (total) order on the complex numbers in the following way: we will say $a + bi < c + di$ iff $a < c$ or ($a = c$ and $b < d$). Does this make $\mathbf{C}$ into an ordered field?

ANSWER: This definition clearly orders $\mathbf{C}$; indeed this manner of ordering any Cartesian product (use the first element unless there is a tie, then move on to the next element to break the tie) is called the *dictionary order* on $\mathbf{R} \times \mathbf{R}$. It's even consistent with addition. But it isn't consistent with multiplication, and indeed there is *no* ordering of $\mathbf{C}$ that is! In any ordered field, squares are positive, and hence sums of squares must be positive, while in $\mathbf{C}$, $i^2 + 1^2 = 0$ is not positive.

3. Show that if $f : X \to Y$ and $g : Y \to Z$ are injections then $g \circ f : X \to Z$ is also an injection.

ANSWER: if $(g \circ f)(x_1) = (g \circ f)(x_2)$ then $g(\, f(x_1)\,) = g(\, f(x_2)\,)$; since g is one-to-one, this means $f(x_1) = f(x_2)$. But f is also one-to-one, so this forces $x_1 = x_2$.

4. Give an example to show that a union of countable sets need not be countable. (Obviously your example must involve infinitely many sets.)

ANSWER: For every real number x, the set $\{x\}$ is obviously countable; but taking the union over all real x will give all of $\mathbf{R}$, which is not a countable set.

If you want to insist that the "countable" sets actually be "countably infinite" (as some people do), then you can use the sets $x + \mathbf{Z} = \{x + n \,|\, n \in \mathbf{Z}\}$ instead of the singletons.

5. Show that $\displaystyle\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n} \right) = \{0\}$

ANSWER: Make sure you prove the inclusions in both directions. (In order to prove "$\subseteq$", you have to use the result that $\mathbf{R}$ does not have any "infinitesimal" numbers, as proved in the text.)

6. Let $X = C[0,1]$, the set of continuous functions $f : [0,1] \to \mathbf{R}$. For f and g in X define $d(f,g) = \displaystyle\int_0^1 |f(x) - g(x)|\, dx$. Show that d defines a metric on X. Which of these two functions is closer to the identity function $f(x) = x$: $g(x) = x^2$ or $h(x) = 1/2$ (constant)?

ANSWER: Most parts of the the definition of a metric are easily met. In order to prove the triangle inequality, note that for any three functions f, g, h and any real number $x \in [0,1]$, the three *numbers* $f(x), g(x), h(x)$ must obey the triangle inequality, so

$$|f(x) - g(x)| \le |f(x) - h(x)| + |h(x) - g(x)|$$

Integrate over all $x \in [0,1]$ to see $d(f,g) \le d(f,h) + d(h,g)$.

It is annoyingly tricky to show that $d(f,g)=0$ can only happen when $f=g$. This is only true because f and g are continuous! But then $f-g$, and also hence the composite $|f-g|$, will be continuous too. It $f \neq g$ then there must be at least one point $x \in [0,1]$ where $(f-g)(x) \neq 0$, and thus $a := |f(x)-g(x)| > 0$. As I mentioned in class, it is then true that for any positive number $b < a$, there will be an interval I around x on which $|f(x')-g(x')| > b$. But then we will have

$$d(f,g) = \int_0^1 |f-g|\,dx > \int_I |f-g|\,dx > \text{width}(I) \cdot b > 0$$

So $d(f,g) > 0$ whenever $f \neq g$.

Finally $d(f,g) = \int_0^1 (x-x^2)\,dx = 1/6$, while $d(f,h)$ can be interpreted as the combined area of two triangles in the graph, which add up to $1/4$. So g is closer to f than h is (using this metric).

7. Let $X = \mathbf{Z}$ and for two different $x, y \in \mathbf{Z}$ define $d(x,y) = 2^{-r}$, where 2^r is the largest power of 2 that divides $x-y$. (When $x=y$ we define $d(x,x)=0$.) Is d a metric on $\mathbf{Z}$?

ANSWER: This *is* a metric on $\mathbf{Z}$, called the "2-adic metric". (There is a p-adic metric defined similarly for any prime p.) The most difficult thing to check seems to be the triangle inequality, but in fact we can easily prove this condition:

$$d(x,y) \leq \min(\ d(x,z), d(y,z)\ )$$

(with inequality holding iff $d(x,z) = d(y,z)$. This is called the "ultrametric condition", and you can prove that this condition actually implies the triangle inequality.

8. Prove the following about all metric spaces X: if x and y are distinct elements of X then there are neighborhoods $N_r(x)$ and $N_s(y)$ around them which are disjoint.

ANSWER: Simply take $r = s = \frac{1}{2}d(x,y)$. Then the disjointness of the two balls follows from the triangle inequality.

It turns out to be handy in Topology if the spaces you are studying have plenty of open sets that can be found to accomplish this sort of thing. How exactly this is phrased will vary by application, but the possible axioms one can impose (like the condition of Problem 8) are called *separation axioms*. The conclusion of Problem 8 (i.e. the ability to separate individual points) is called the *Hausdorff Property*. There are other separation axioms one could impose, such as the ability to separate any point from any (other) closed set, or the ability to separate any two disjoint closed sets, etc.