M365C (Rusin) HW2 – due Thursday, Sept 12 2019

1. Show that the intervals $(0, 1)$ and $(0, \infty)$ have the same cardinality.

2. If $f : X \to Y$ is a function and $A \subseteq X$ and $B \subseteq Y$, we can define sets

$$f(A) = \{f(x) \,|\, x \in A\} \subseteq Y \qquad \text{and} \qquad f^{-1}(B) = \{x \in X \,|\, f(x) \in B\} \subseteq X$$

Pick at least three of the following statements and either prove it must be true or give a counterexample.

$$\begin{array}{cc} f(A_1 \cap A_2) = f(A_1) \cap f(A_2) & f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2) \\ f(A_1 \cup A_2) = f(A_1) \cup f(A_2) & f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2) \\ f(f^{-1}(B)) = B & f^{-1}(f(A)) = A \end{array}$$

(When counterexamples exist, you might want to ask yourself whether some additional conditions might be imposed on f, X, A etc to make the statement true!)

3. I said in class that no matter how big a set is, its power set is always bigger. Prove this: if X is any set and $f : X \to \mathcal{P}(X)$ is any function, then f is not a surjection. (Hint: $A = \{x \in X \,|\, x \notin f(x)\}$ is an element of $\mathcal{P}(X)$. You might get less of a headache if you first experiment with some fs defined on some small sets X!)

4. If X and Y are ordered sets, and $f : X \to Y$, then we say that f is an *increasing function* if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. Prove that an increasing function $f : \mathbf{N} \to \mathbf{N}$ necessarily has $x \leq f(x)$ for all $x \in \mathbf{N}$. (Use induction.)

5. Show that the only function $f : \mathbf{R} \to \mathbf{R}$ which preserves both addition and multiplication is the identity. (That is, if for every $x, y \in \mathbf{R}$ we have $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, then $f(z) = z$ for all $z \in \mathbf{R}$.) Hint: show $f(z) \geq 0$ whenever $z \geq 0$ by thinking about squares; then show f is increasing. On the other hand, show $f(z) = z$ whenever z is an integer, and then whenever z is rational. Then complete the proof.

6. Let $F = \mathbf{Z}_3 \times \mathbf{Z}_3$. Define two operations on F as follows:

$$(a, b) + (c, d) = (a + c, b + d) \qquad \text{and} \qquad (a, b) \cdot (c, d) = (ac - bd, bc + ad)$$

where the operations on the right sides of the equal signs are the arithmetic in $\mathbf{Z}_3$. Use the exact same formulas to define "addition" and "multiplication" operations on $G = \mathbf{Z}_5 \times \mathbf{Z}_5$. One of F and G is a field, and the other is not. Which is which, and why?