M365C (Rusin) HW3 – due Thursday, Sept 19 2019

1. Compute $\inf(S)$ and $\sup(S)$ where $S = \left\{ \frac{1}{n+1} - \frac{1}{m+1} \ : \ m, n \in \mathbf{N} \right\}$

2. Define a (total) order on the complex numbers in the following way: we will say $a + bi < c + di$ iff $a < c$ or ($a = c$ and $b < d$). Does this make $\mathbf{C}$ into an ordered field?

3. Show that if $f : X \to Y$ and $g : Y \to Z$ are injections then $g \circ f : X \to Z$ is also an injection.

4. Give an example to show that a union of countable sets need not be countable. (Obviously your example must involve infinitely many sets.)

5. Show that $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n} \right) = \{0\}$

6. Let $X = C[0,1]$, the set of continuous functions $f : [0,1] \to \mathbf{R}$. For f and g in X define $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$. Show that d defines a metric on X. Which of these two functions is closer to the identity function $f(x) = x$: $g(x) = x^2$ or $h(x) = 1/2$ (constant)?

7. Let $X = \mathbf{Z}$ and for two different $x, y \in \mathbf{Z}$ define $d(x,y) = 2^{-r}$, where 2^r is the largest power of 2 that divides $x - y$. (When $x = y$ we define $d(x,x) = 0$.) Is d a metric on $\mathbf{Z}$?

8. Prove the following about all metric spaces X: if x and y are distinct elements of X then there are neighborhoods $N_r(x)$ and $N_s(y)$ around them which are disjoint.