M365C (Rusin) HW8 – due Thursday, Oct 31 2019

1. This question continues the investigations of Problem 2 on HW7.

If $\{f_n(x)\}$ is a sequence of functions $f_n : X \to \mathbf{R}$ with the same domain and codomain, we say the sequence *converges pointwise* to $f : X \to Y$ if for every $x \in X$, the numbers $\{f_n(x)\}$ converge to the number $f(x)$. (To understand this, we of course need the metric in $\mathbf{R}$ but X doesn't even have to be a metric space, just a set; and we don't need to discuss any sort of "distance between functions".)

Now, there is also the concept of *uniform convergence*. We say that a sequence of functions $f_n : X \to \mathbf{R}$ converges uniformly to $f : X \to \mathbf{R}$ if:

$$\forall \epsilon > 0\ \exists N \in \mathbf{Z}\ \forall x \in X\ \forall n > N \text{ we have } |f_n(x) - f(x)| < \epsilon$$

(The only difference between this and pointwise convergence is that now the phrase "there is an integer N such that" comes *before* the phrase "for every $x \in X$".)

Show that $\{f_n\}$ converges uniformly to f if and only if $d_\infty(f_n, f)$ converges to 0. You may assume that $X = [0, 1]$ if you like, so that you are proving that uniform convergence is the same as convergence in the metric space $(C^0[0, 1], d_\infty)$.

2. For any function $f : \mathbf{R} \to \mathbf{R}$ and any $a \in \mathbf{R}$ define

$$f^*(a) = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{h}$$

(a) If f is differentiable at a, evaluate $f^*(a)$.
(b) If $f^*(a)$ exists, must f be differentiable at a?

3. Suppose

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

Show that f is differentiable at $x = 0$ and compute $f'(0)$.

Bonus: This will mean that $g(x) := f'(x)$ is defined for all real x. Is g differentiable at 0?

4. We say that a function $f : \mathbf{R} \to \mathbf{R}$ is *increasing on* a subset S of $\mathbf{R}$ if:

$$\text{for all } x, y \text{ in } S, \text{ if } x < y \text{ then } f(x) < f(y)$$

(a) Prove that if f is differentiable on an interval (a, b) then f is increasing on (a, b) iff $f'(x) > 0$ for all $x \in (a, b)$.
(b) Is $f(x) = 1/x$ increasing on $S = \mathbf{R} - \{0\}$?

5. Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable at every point $x \in \mathbf{R}$ and moreover that for each x, $|f'(x)| < 5$. Show that f is uniformly continuous on $\mathbf{R}$.