M365C (Rusin) TEST 1 ANSWERS — Tuesday, Oct 8 2019

Comment: The test was too long, don't you think? There will surely be a curve.

1. (a) Prove the following: for all $a, b \in \mathbf{R}$ with $a < b$ there exists a rational multiple of $\sqrt{2}$ between a and b (that is, a number of the form $c\sqrt{2}$ with $c \in \mathbf{Q}$.

(b) Give an example of a subset of $\mathbf{R}$ which is neither open nor closed nor bounded nor compact (and explain why it fails to have each of those properties).

ANSWER: (a) We know that there is a rational number c between $a/\sqrt{2}$ and $b/\sqrt{2}$ since $\bar{\mathbf{Q}} = \mathbf{R}$. Then $a < c\sqrt{2} < b$.

(b) Something like $A = [0, 1.5) \cup (2, \infty)$ will do. It's not open because every ball $N_r(0)$ includes some negative numbers. It's not closed because 1.5 is a limit point not in A. (Every ball $N_r(1.5)$ includes elements of A). It's not bounded because it includes $\mathbf{N}$ which we know is unbounded. It's not compact because every compact set would have to be both closed and bounded.

2. Show that the set of all *finite* sets of natural numbers is countable. (Hint: every finite set of natural numbers has a largest element.)

ANSWER: All we need to know is the following

LEMMA: For every $n \in \mathbf{N}$, the set

$$S_n = \{A \subseteq \mathbf{N} \,|\, max(A) = n\}$$

is finite.

Proof: The elements of S_n are of the form $\{n\} \cup B$ for $B \in \mathcal{P}(\{0, 1, 2, \ldots, n-1\})$, so $|S_n| = 2^n$.

Then we can enumerate all the finite subsets of $\mathbf{N}$ by listing first $\emptyset$, then the elements of S_0, then the elements of S_1, etc.

Alternatively, you can map every finite set of $\mathbf{N}$ to a positive integer by sending the set $\{a_1, a_2, \ldots, a_n\}$ (with the elements listed in ascending order) to the integer $2^{a_1} 3^{a_2} 5^{a_3} \ldots$, the bases $2, 3, 5, \ldots$ being the primes in order. Because of the Fundamental Theorem of Arithmetic, this mapping is one-to-one, so the set of all finite sets of natural numbers is paired with a subset of the collection of positive integers, and hence is countable.

3. In Homework 3 we defined a (total) order on the set of complex numbers in the following way: we said $a + bi < c + di$ iff $(a < c)$ or $(a = c$ and $b < d)$. Find a subset of the complex numbers which has upper bounds in this sense, but has no *least* upper bound.

ANSWER: You may for example use $A = \{a + bi \,|\, a < 0\}$. Every element in A is less than $0 + 0i$, so the set is bounded, but if $a + bi$ is an upper bound, so is the smaller number $a + (b - 1)i$ (since $z + i \in A$ whenever $z \in A$). So there is no *least* upper bound.

4. Suppose X is a metric space with metric d, and suppose $x_0 \in X$. Let $C = \{x \in X \mid d(x, x_0) \leq 1\}$. Show that C is a closed set. (Hint: complements)

Extra Credit: in the special case of $X = \mathbf{Z}$ with the 2-adic metric d, with $x_0 = 0$, show that C is also open!

ANSWER: The complement of C is the set of points in X whose distance from x_0 is strictly greater than 1. If $x \in X$ lies in this complement $X - C$, then let $r = d(x, x_0) > 1$. Then I claim the whole ball $B = N_{(r+1)/2}(x)$ lies in $X - C$. Indeed, if $b \in B$ then $d(b, x) < (r + 1)/2$ while by the triangle inequality $d(x, x_0) \leq d(x, b) + d(b, x_0)$ and so $d(b, x_0) \geq r - (r + 1)/2 = 1 + (r - 1)/2 > 1$, meaning $b \in X - C$. Since this is true for every $b \in B$, $B \subseteq X - C$. Thus x lies in the interior of $X - C$. Since $x \in X - C$ was arbitrary, $X - C$ is open, and thus C is closed.

In the 2-adic case, this particular closed ball is the whole set $\mathbf{Z}$, which is open. I had actually intended to ask you a bit more so let me spell it out here: in this metric space, distances can only be of the form 2^{-r} for some natural number r. That's a discrete set of possible values for the distance function. So saying that a distance is less than or equal to some value is equivalent to saying that that distance is less than the next possible value. As a consequence, every closed ball is *also* an open ball! (And vice versa!)

5. Show that this sequence converges, and determine its limit:

$$2,\ 2 + \frac{1}{2},\ 2 + \frac{1}{2 + \frac{1}{2}},\ 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}},\ \ldots$$

(Hint: to prove convergence, show that the two subsequences consisting of every other term are monotomic.)

ANSWER: First note that the terms of the sequence x_n have $x_0 = 2$ and $x_n = 2 + 1/x_{n-1}$ whenever $n > 0$. So *if* the sequence converges, to some nonzero limit L, then $2 + 1/x_{n-1}$ would converge to $2 + 1/L$, and we would have $L = 2 + 1/L$, i.e. $L^2 - 2L - 1 = 0$. That means $L = 1 \pm \sqrt{2}$. Obviously all the terms of the sequence are positive — indeed they are all larger than 2 — so the sequence cannot converge to 0 nor to $1 - \sqrt{2}$. So the sequence will converge to $1 + \sqrt{2}$ if it converges at all.

There are several ways to prove convergence. Perhaps the simplest is to note that if $x_n > L$ then $x_{n+1} = 2 + 1/x_n < 2 + 1/L = L$ and vice versa, that is, the sequence will alternate between the two sides of L. So let $a_n = x_{2n}$ and $b_n = x_{2n+1}$. Each a_n is less than L, and each b_n is greater than L. Thinking about the graph of the parabola allows us to phrase this is:

$$a_n^2 - 2a_n - 1 < 0 < b_n^2 - 2b_n - 1$$

for every n.

Very well: how far do these sequences move?

$$a_{n+1} - a_n = 2 + \frac{1}{2 + \frac{1}{a_n}} - a_n = \frac{5a_n + 2}{2a_n + 1} - a_n = (-2)\frac{a_n^2 - 2a_n - 1}{2a_n + 1}$$

But that quadratic numerator is negative for every n, meaning $a_{n+1} - a_n > 0$: the as increase, while bounded above (by L). With exactly the same algebra we deduce that the bs decrease and are bounded below (by L).

So both sequences do converge. As in the first paragraph we let L' be the limit of the a_n and then take limits of the the sequences in the previous paragraph: $0 = L' - L' = (-2)(L'^2 - 2L' + 1)$, so L' is a root of the same quadratic that defined L; so is $L'' = \lim(b_n)$. Since all are positive, we find $L' = L''$, so the original sequence x_n must converge to this same value.

6. I asked my calculus students to decide whether the series

$$\sum_{n\geq 1} \frac{(n!)^2}{(2n)!} \left(\frac{2}{5}\right)^n$$

converges. Sadly, some of them don't seem to know the difference between that series and the product

$$\left(\sum_{n\geq 1} \frac{(n!)^2}{(2n)!}\right) \cdot \left(\sum_{n\geq 1} \left(\frac{2}{5}\right)^n\right)$$

But that prompts the following exercise: show that if $\sum a_n$ and $\sum b_n$ are both convergent series of *positive* real numbers, then the series $\sum(a_n b_n)$ converges.

ANSWER: Since all the terms are given to be positive, the partial sums form an increasing sequence, and we need only find an upper bound for them.

Perhaps the easiest way is to first note that for every N,

$$(a_1 b_1) + (a_2 b_2) + \ldots + (a_N b_N) \leq (a_1 + a_2 + \ldots + a_N)(b_1 + b_2 + \ldots + b_N)$$

because the left side is just what remains after expanding the right side and dropping a few terms. Then note that the convergence of $\sum a_n$ (and the positivity of the a_n) means the partial sums of this series are bounded by the infinite sum; and likewise for the b_n. So we conclude

$$(a_1 b_1) + (a_2 b_2) + \ldots + (a_N b_N) \leq (\sum a_n)(\sum b_n)$$

which is finite. So there's an upper bound for the partial sums, and thus they converge (monotonically).

Positivity is necessary here: if $a_n = b_b = (-1)^n/\sqrt{n}$ then both $\sum a_n$ and $\sum b_n$ converge, but $\sum(a_n b_n)$ does not.