Students: You asked for some detailed solutions to the problems from HW6 to study for the test. I have a few minutes so here you go!

Problem 1g (section 3.3) asked for the solution of

$$\varepsilon y'' + 2y' + e^y = 0, \qquad y(0) = y(1) = 0$$

This example is *not* amenable to the basic pertubation approach, in which we would try to find a power series solution which, for $\varepsilon = 0$, reduces to the solution of the corresponding differential equation. Because: there *is* no solution when $\varepsilon = 0$! The equation $2y' + e^y = 0$ has the general solution $y = -\log(x/2 + C)$ and if we take $C = 1/2$ we get the one solution

$$y_r = -\log((x+1)/2)$$

which is consistent with the boundary condition at the right-hand endpoint $x = 1$. But it has the value $\log(2)$ at $x = 0$ and so the original problem has no solution at all when $\varepsilon = 0$. On the other hand, we can see graphically that something like this will work if ε is tiny enough: simply follow this curve from $x = 1$ back *almost* to $x = 0$, then head down to the origin to match the other boundary condition; obviously such a change requires very large slopes for small x but we can reasonably expect those to be balanced by the even-larger curvature required.

So this is the "boundary layer": for x close to 0 we expect y between 0 and $\log(2)$, so e^y is between 1 and 2, which will be negligible compared to the other two terms, which we just decided graphically should be very large. To make this fuzzy thinking precise, we rescale x, turning the smallest x into moderately-sized values of $\bar{x} = x/\varepsilon$. (That's a little bit of a guess; sometimes we find that setting $\bar{x}$ equal to say $x/\sqrt{\varepsilon}$ or x/ε^2 works better in the next step).

So let's see how the ODE looks, expressed in terms of this variable. There's no reference to x itself in the original equation, but now we will not speak of $y' = dy/dx$ but rather of $\dot{y} = dy/d\bar{x} = (dy/dx)\varepsilon$. The equation relatinng these terms is

$$\varepsilon \frac{\ddot{y}}{\varepsilon^2} + 2\frac{\dot{y}}{\varepsilon} + e^y = 0$$

This captures the idea of the previous paragraph: y' and y'' are huge but will have to nearly cancel because e^y is not huge. Clearing denominators we get $\ddot{y} + 2\dot{y} + \varepsilon e^y = 0$ (and $y = 0$ at both $\bar{x} = 0$ and $\bar{x} = 1/\varepsilon$).

This time we *can* obtain solutions by regular pertubation: when $\varepsilon = 0$ we have the first integral $\dot{y} = Ae^{-2\bar{x}}$ and so the general solution vanishing at $\bar{x} = 0$ is $y_0 = B(1 - e^{-2\bar{x}})$ for any constant B. There really isn't a second boundary condition to satisfy because there really isn't a second boundary! (You might hope that y should die off to zero as $\bar{x} \to \infty$, but that's not possible for any B.) If we wish, we could extend this family of solutions to include nonzero values of ε, adding $y_1\varepsilon + y_2\varepsilon^2 + \ldots$. But I won't pursue this because it involves evaluating some nasty integrals, and introduces an unknown constant for each of y_1, y_2, etc. (Stated another way, our unknown "constant of integration", the part of the general solution not pinned down because we lack a second boundary condition or second initial condition, is not really a number now but rather a function of ε, i.e. an infinite list of numbers, those being the coefficients of the Taylor series of this function of ε.)

In any event, we now have a family of functions which suggest the behaviour of y near $x = 0$, at least for small ε: we expect

$$y = y_\ell(x) = B(1 - e^{-2x/\varepsilon})$$

Now, which value of B is appropriate? We know that for small ε we already expect $y \approx \log(2)$ for small x, so we should choose B so that y_ℓ is about this large for some combinations of small x and small ε. We might for example take $B = \log(2)/(1 - e^{-2})$ so that the graphs of $y\ell$ and y_r roughly glue together at $x = \varepsilon$ to give a continuous function.

If you had read the book before tackling the assignment, you found an alternative perspective: you can remove yourself from the analysis, along with your idiosyncratic choice of where to end the "boundary later". That's good but the recipe they give (for the "unform" approximation, also called the "method of matched expansions") has the effect that it doesn't quite satisfy the differential equation nor the boundary condition! But it will give good agreement with both partial solutions on the region where both should be valid. They do this by adding the solutions y_ℓ and y_r, and subtract the value on the overlapping region, with the B chosen so that the functions agree at "opposite ends": we want

$$L = \lim_{x \to 0} y_r(x) = \lim_{\bar{x} \to \infty} y_\ell(\bar{x})$$

This requires $B = \log(2)$, and then we let $y(x) = y_\ell(x) + y_r(x) - L$, i.e.

$$y(x) = -\log((x+1)/2) + \log(2)(1 - e^{-2x/\varepsilon}) - \log(2) = -\log((x+1)/2) - \log(2)e^{-2x/\varepsilon}$$

You may wish to compare this asymptotic solution to the solutions obtained numerically with your favorite software, for example when $\varepsilon = 0.1$. You will see that the approximation we have created is always a little low — and in particular takes the wrong value at the right-hand endpoint — but the error is never more than about 0.02, and is worst near the middle. But overall, this approximation captures the general shape of the numerically-correct solution: it starts at $y = 0$, rises to a peak of about $y = 0.54$ near $x = 0.14$ (which is to say as we pass through the boundary layer), and then descends to near $y = 0$ at the right endpoint.

Problem 7 is a little unusual because the right-hand boundary is infinitely far away. From the perspective of complex analysis, this is not really a different situation ("∞" is just another point on the Riemann Sphere!) and anyway you can make ∞ look like just another real number by performing a change of variables like $x' = 1/(1 + x)$, say, transforming our domain to the interval $(0, 1)$. But it's not really a problem to treat this ODE as is:

$$\varepsilon y'' + e^x y' = 1, \qquad y(0) = 1, \quad y(\infty) = 0.$$

When $\varepsilon = 0$ the general solution is $y(x) = C - e^{-x}$ and if $C = 0$ this function satisfies the right-hand boundary conditions. It's obviously inapplicable at the left edge because this function is negative everywhere! You can improve the situation a bit if you like, using pertubation analysis: the solution of the original ODE which meets the right-hand boundary condition is $y_r(x) = -e^{-x} - \frac{\varepsilon}{2}e^{-2x} + \ldots$*

On the other end we can look for transient behaviour on a boundary layer. The scaling $\bar{x} = x/\varepsilon$ renders the ODE as

$$\ddot{y} + e^{\varepsilon\bar{x}}\dot{y} = \varepsilon, \qquad y(\bar{x}=0) = 1$$

This can be solved at $\varepsilon = 0$, too; clearly we first get $\dot{y} = Ae^{-\bar{x}}$ and then $y = 1 + A(1 - e^{-\bar{x}})$ on account of the initial condition.

Now we choose the A to accomplish matching between the solutions that work near the two ends. The right-hand solution $y = -e^{-x}$ tends, as we approach the left-hand region, to $y = -1$; the left-hand solution $y = 1 + A(1 - e^{-x/\varepsilon})$ tends to $1 + A$ quickly as we approach the right-hand region. If we take $A = -2$ then these two solutions will be approximately equal away from the two endpoints, and near either endpoint one function stays nearly constant while the other exhibits its own behaviour. So we can capture the intermediate values and both end-point behaviours with the function

$$y = y_\ell + y_r - L = \left(1 - 2(1 - e^{-x/\varepsilon})\right) + \left(-e^{-x}\right) - (-1) = 2e^{-x/\varepsilon} - e^{-x}$$

One reason I have not emphasized the "uniform approximation" is that in general it does not provide a function meeting the boundary-value conditions. If we have found solutions y_ℓ and y_r that provide approximate solutions to an ODE near the endpoints of an interval $[a, b]$, we would like it if

$$\lim_{x\to b^-} y_\ell(x) = \lim_{x\to a^+} y_r(x) = L,$$

say, for then the final proposed solution $y(x) = y_\ell(x) + y_r(x) - L$ will approach $y_\ell(a)$ on the left and $y_r(b)$ on the right; if we've already made those two functions match one boundary condition each, then $y(x)$ itself will match both of them. If y_ℓ is nearly constant near $x = b$ then y will have the same behaviour as y_r there (increasing or decreasing, concave or convex, oscillatory, etc.) Likewise if y_r is nearly constant near $x = a$. So the idea of choosing parameters so that the appropriate limits match (i.e. are equal to a single number L) is a sound one when it's the behaviour near the endpoints that's tricky.

Unfortunately that gets a little complicated in some cases like e.g. Problem 1g here: the left solution y_ℓ was presented as a function of $\bar{x} = x/\varepsilon$ and so rather than taking $\lim_{\bar{x}\to 1/\varepsilon}$, we took $\lim_{\bar{x}\to\infty}$ since, after all, our function y_ℓ was chosen to satisfy the ODE only when $\varepsilon = 0$. The whole business is a little tricky to put onto a solid foundation: one must make assumptions not only about y staying nearly constant but also about y' and y'' since, obviously, they are involved in the ODE; and the quality of the approximation

* Actually, the whole series is $-\sum \varepsilon^{n-1}(n-1)!e^{-nx}/n$, a series which is not convergent for any x! One must not assume that adding additional terms necessarily improves the approximation for any particular x.

depends on both x and ε, and it's not immediately clear what "small" means and so on. This is not a course in which we are trying to give solid proofs of the robustness of such techniques. Rather, the idea is "try it and see what happens": of the multiple matching techniques available to you, pick one and see whether the results seem appropriate to the application. If not, either try incremental adjustments (e.g. adding terms of a Taylor series in ε) or try switching to a different method altogether (e.g. making a piece-wise smooth function from y_ℓ and y_r rather than making the uniform approximation from their sum.)