I asked you to solve the differential equation

$$y'' + y = \varepsilon y(y')^2$$

Of course I intended for you to think of this as a pertubation of the simpler ODE $y''+y=0$ which turns out to require re-scaling time ($\tau = \omega t$) as well. But it's also interesting to compare this answer to the answers that can be obtained other ways. Allow me to show you what kinds of tools you already have to analyze a differential equation!

This is an autonomous second-order equation so we can view the problem for any fixed ε as a differential system

$$y' = v \qquad v' = -y + \varepsilon y v^2$$

I invite you to sketch out the orbits in the $y-v$ plane to get a qualititative understanding of the motion on a line (y=position, v=velocity). If $\varepsilon > 0$ there are orbits with v constant (at $v = \pm 1/\sqrt{\varepsilon}$); outside them the trajectories look vaguely parabolic. Between the two lines the orbits are closed loops circulating around the origin (which is also an orbit of its own). We can interpret these trajectories physically: the origin is the solution to the ODE corresponding to an unmoving "oscillator" ($y(t) = 0$ for all times t) and the solutions with constant v correspond to simple linear motion ($y(t) = mt + b$ where the speed $|m|$ must be exactly $1/\sqrt{\varepsilon}$). The distant trajectories are motion along a line that involve racing at high speed towards the origin $y = 0$, slowing down a little, and then accelerating away from the origin. It is the orbits nearer to the origin in $\mathbf{R}^2$ that correspond to interesting oscillatory behaviour. These are precisely the orbits that include a moment at which the object is briefly at rest ($v = 0$); as you might expect, starting further from the origin involves a larger oscillation, eventually achieving a higher speed $|v|$ (the maximum speed occurs as the object passes the origin $y = 0$, but if $\varepsilon > 0$ that speed is never larger than $1/\sqrt{\varepsilon}$).

As it turns out, we can more or less solve this ODE in closed form as well. To do this, we first describe those trajectories (above) algebraically. The differential system is autonomous, so we may simply divide one equation by the other and use the chain rule to conclude

$$\frac{dy}{dv} = \frac{-v}{y(1-\varepsilon v^2)}$$

a separable differential equation whose general solution is $\log(|1-\varepsilon v^2|) = \varepsilon y^2 + C$ i.e. each of the curves in the previous picture is the solution set to

$$1 - \varepsilon v^2 = K e^{\varepsilon y^2}$$

for some constant K; when the motion is oscillatory, there is a point $(y, v) = (y_0, 0)$ on the curve, in terms of which we may express K and then solve for velocity v:

$$v = \pm\sqrt{\frac{1 - e^{\varepsilon(y^2 - y_0^2)}}{\varepsilon}}.$$

In general this forces $|v|$ to be just a bit less than $1/\sqrt{\varepsilon}$ except when y is very near the edges of the loop ($y \approx \pm y_0$).

Of course v is the velocity $\frac{dy}{dt}$ of the motion, so we can view this explicit form of v as a differential equation to be solved for $y = y(t)$, subject to the initial condition that $y(0) = y_0$ (where I have defined the moment $t = 0$ to be the time when the particle has reached its greatest displacement along the line and hence its instantaneous velocity is zero).

Well! This ODE is also separable; in fact we can compute what time it is (t) from our position y:

$$t = \int_{u=y}^{u=y_0} \sqrt{\frac{\varepsilon}{1 - e^{\varepsilon(u^2 - y_0^2)}}}\, du$$

In particular, we can compute the length of time needed for one period of the oscillation; that's four times the length of time needed to travel from $y = y_0$ to $y = 0$, so we finally conclude with the precise formula for the period in our problem:

$$\text{Period} = 4 \int_{u=0}^{u=1} \sqrt{\frac{\varepsilon}{1 - e^{\varepsilon(u^2 - 1)}}}\, du$$

This integral looks horrible, of course, but it can be evaluated numerically for any particular ε. Alternatively, we can look to see how it depends on ε. Write the integrand as

$$\frac{1}{\sqrt{1-u^2}} \left(\frac{X}{e^X - 1} \right)^{1/2} \qquad \text{where} \qquad X = -\varepsilon(1 - u^2)$$

and then expand that function of X in a Taylor Series, say $\sum_{k \geq 0} \alpha_k X^k$. Then the integrand is $\sum_{k \geq 0} \alpha_k (-1)^k (1 - u^2)^{k - \frac{1}{2}} \varepsilon^k$ which we can integrate term-by-term using

$$\int_0^1 (1 - u^2)^{k - 1/2}\, du = \frac{\pi}{2} \binom{2k}{k} \frac{1}{4^k} \qquad (k = 0, 1, 2, \ldots)$$

and so we obtain a formula for the period of the oscillation:

$$\text{Period} = 2\pi \sum_{k \geq 0} \alpha_k (-1)^k \binom{2k}{k} \frac{1}{4^k} \varepsilon^k$$

I cheated and used a bit of computer-algebra help to find the expansion *

$$\left(\frac{X}{e^X - 1} \right)^{1/2} = 1 - \frac{1}{4} X + \frac{1}{96} X^2 + \frac{1}{384} X^3 - \frac{1}{10240} X^4 + \ldots$$

* It's an intriguing power series: the terms α_k seem to alternate in sign but it's *two* positives followed by *two* negatives. Also the numbers $4^k (k+1)! \alpha_k$ are nearly integral, although I don't recognize the numerators.

to which we then apply the formula above for the definite integrals to conclude

$$\text{Period} = 2\pi\left(1 + \frac{1}{8}\varepsilon + \frac{1}{256}\varepsilon^2 - \frac{5}{6144}\varepsilon^3 - \frac{7}{262144}\varepsilon^4 + \dots\right)$$

or, to tie back to the substitution you used yourself,

$$\text{Period} = 2\pi/\omega \quad \text{where} \quad \omega = \left(1 - \frac{1}{8}\varepsilon + \frac{3}{256}\varepsilon^2 - \frac{1}{6144}\varepsilon^3 - \frac{79}{786432}\varepsilon^4 + \dots\right)$$

By the way, the integral giving the period is valid for any ε, including large ones. As ε grows large, the two lines in the phase diagram we discussed earlier come closer together, making our starting position $(y_0, v_0) = (1, 0)$ feel "far" from the origin, so the orbit will spend most of its time near the straight lines, meaning the speed will stay nearly constant, almost equal to the maximum value of $1/\sqrt{\varepsilon}$. This will mean that the time needed to complete the orbit should be nearly proportional to $\sqrt{\varepsilon}$. This is borne out numerically; for example when $\varepsilon = 1000$, the period is about 20.14564250; when $\varepsilon = 100,000$ the period is about 201.3168484. (As $\varepsilon \to -\infty$, the period decreases to zero, roughly proportional to $1/\sqrt{|\varepsilon|}$.)

In any event once we know the period of the oscillation, we can rescale time by letting $\tau = \omega t$ with ω as above, so that y can be viewed as a function of τ that repeats precisely when τ runs over an interval of length 2π. Using dots to indicate derivatives with respect to τ, our differential equation may then be written

$$\ddot{y} + \mu y = \varepsilon y(\dot{y})^2 \quad \text{where} \quad \mu = 1/\omega^2 = 1 + \frac{1}{4}\varepsilon + \frac{3}{128}\varepsilon^2 - \frac{1}{1536}\varepsilon^3 - \frac{95}{393216}\varepsilon^4 - \dots$$

You may now express y as a pertubation series $y = y_0 + y_1\varepsilon + y_2\varepsilon^2 \dots$ and solve for each y_i in turn; no "secular terms" should arise because we have already scaled time appropriately.

Alternatively, you may use the fact that y is periodic and as a function of τ has period 2π to express y as a Fourier series, $y(\tau) = \sum(a_n \sin(n\tau) + b_n \cos(n\tau))$ for some constants a_n and b_n (which will depend on ε). You have already seen how the lowest-order (in ε) parts of the solution work: our a_n will all be zero and the coefficients b_n will be power series in ε which start with increasingly large powers of ε. I worked out the first few terms this way:

$$\begin{array}{lllllll}
y(t) = & \cos(\tau) & (1 & -1/32\varepsilon & +5/3072\varepsilon^2 & +9/32768\varepsilon^3 & -913/23592960\varepsilon^4 & +\dots) \\
& +\cos(3\tau) & (& 1/32\varepsilon & -1/256\varepsilon^2 & -1/16384\varepsilon^3 & +75/1048576\varepsilon^4 & +\dots) \\
& +\cos(5\tau) & (& & 7/3072\varepsilon^2 & -31/73728\varepsilon^3 & -19/4718592\varepsilon^4 & +\dots) \\
& +\cos(7\tau) & (& & & 61/294912\varepsilon^3 & -13/262144\varepsilon^4 & +\dots) \\
& +\cos(9\tau) & (& & & & 109/5242880\varepsilon^4 & +\dots)
\end{array}$$

Read in rows to see how each $\cos(n\tau)$ component varies with ε; read in columns to get the Fourier expansion of each $y_i(\tau)$. For small ε there are only a couple of cosines that are significant but for say $\varepsilon = 10$ the Fourier expansion is something like

$$y(t) = .860\cos(\tau) + .089\cos(3\tau) + .028\cos(5\tau) + .012\cos(7\tau) + .005\cos(9\tau) + .003\cos(11\tau)$$

whose graph is more of a zig-zag (reflecting the behaviour seen in the phase-diagram: the oscillator spends most of its time traveling with speed near $1/\sqrt{\varepsilon}$).

See? A simple differential equation like

$$y'' + y = \varepsilon y(y')^2$$

can convey a lot of meaning that we have the tools to discover!