Math 408D (Rusin) — EXAM 3 — Nov 23 2010. **ANSWER KEY**

1. Find an equation for the plane passing through the points

$$A = (1,2,3), \qquad B = (4,5,6), \qquad C = (1,0,2)$$

We compute the vectors $AB = \langle 3,3,3\rangle$ and $CA = \langle 0,2,1\rangle$ which lie in the plane; thus the vectors normal to the plane must be perpendicular to both $u = \langle 1,1,1\rangle$ and $v = \langle 0,2,1\rangle$. You can compute such vectors by insisting that two dot products be zero, but it's easier to compute $u \times v = \langle -1,-1,2\rangle$. Thus the plane is perpendicular to the vector $\langle 1,1,-2\rangle$ and so has the form $x + y - 2z =$constant. Trying any of the three given points shows that the constant must be 5.

You could also have started simply from the premise that the equation of the plane would be of the form $ax + by + cz = d$; since the equation must be satisfied at each of the three points, you get three equations to solve, from which a, b, c can be determined (in terms of d).

2. Two particles are traveling through space; their positions at time t are respectively

$$r_1(t) = (t, t^2, t^3) \qquad \text{and} \qquad r_2(t) = (1 + 2t, 1 + 6t, 1 + 14t).$$

Do the particles collide? If so, where? Do there paths intersect? If so, where?

They don't collide: they only have the same x-coordinates when $t = -1$, but at that moment they are at locations $(-1,1,-1)$ and $(-1,-5,-13)$ respectively.

But their paths do intersect. They can both arrive at point P (at different times t_1 and t_2, say) if $P = r_1(t_1) = r_2(t_2)$, that is, if there is a combination of t_1 and t_2 which makes

$$(t_1, t_1^2, t_1^3) = (1 + 2t_2, 1 + 6t_2, 1 + 14t_2)$$

Comparing the x coordinates shows $t_1 = 1 + 2t_2$; but then the y-coordinates will only be equal if $(1 + 2t_2)^2 = 1 + 6t_2$, i.e. if $4t_2^2 = 2t_2$. This equation holds iff $t_2 = 0$ or $t_2 = 1/2$. We still need the z-coordinates to be equal too, but indeed $r_2(0) = (1,1,1) = r_1(1)$ and $r_2(1/2) = (0,0,0) = r_1(0)$, that is, the paths cross at these two points.

3. A butterfly flies through space, starting at time $t = 0$. Its position at time t is

$$(e^t)\ \mathbf{i} + (e^t \sin t)\ \mathbf{j} + (e^t \cos t)\ \mathbf{k}$$

How far has it traveled by time t?

The butterfly's velocity is $\langle e^t, e^t(\cos t + \sin t), e^t(\cos t - \sin t)\rangle$ from which we compute its speed to be $[(e^t) + (e^t(\cos t + \sin t))^2 + (e^t(\cos t - \sin t))^2]^{1/2}$ which works out to $\sqrt{3}e^t$. The distance the butterfly has traveled by time t is then the integral of this from time 0 to time t, namely $\sqrt{3}(e^t - 1)$.

4. Mr. Wile E. Coyote has purchased a cannon from the Acme corporation. The cannon is capable of shooting a ball with an (initial) speed of 70m/s. There is a certain bird which Mr Coyote wishes to hit; the bird is exactly 400m away. How should he aim the cannon? (For this problem you may assume they are both on level ground, the bird does not move, and the cannon will not change its angle of elevation after Mr Coyote aims it...)

Setting up coordinates so that x measures the horizontal distance from the coyote towards the bird, and y measure the vertical distance up from the ground (both in meters), we know from Physics that the ball's acceleration will be a constant vector: $a(t) = (0, -9.8)$. Thus the velocity has the form $v(t) = (c_1, -9.8t + c_2)$ for some constants c_1 and c_2; this will mean the initial velocity will be $v(0) = (c_1, c_2)$, so if the cannon is aimed to shoot at an angle of θ off the horizontal, we must have $c_1 = 70\cos\theta$ and $c_2 = 70\sin\theta$. Finally then the position at time t is of the form $p(t) = (c_1t + c_3, -4.9t^2 + c_2t + c_4)$ for some constants c_3 and c_4; but our choice of coordinates puts the cannon at the origin, so we must have $p(0) = (0, 0)$ and that means $c_3 = c_4 = 0$.

So, great! We know where the ball is at every moment t: its location is

$$(70(\cos\theta)t, 70(\sin\theta)t - 4.9t^2)$$

In particular, we know when it's at the ground: that happens when $y = 0$, i.e. when $70(\sin\theta)t = 4.9t^2$, which means either $t = 0$ (the moment of launch) or $t = 70(\sin\theta)/4.9$. So that latter is the moment of impact, at which time the ball's location is $(70(\cos\theta)t, 0) = (70(\cos\theta) \cdot 70(\sin\theta)/4.9, 0)$

Now, we want to choose θ so that this location is where the bird is: $(400, 0)$. Thus the right values of θ are those with $4900\sin\theta\cos\theta = 400 \cdot 4.9$, i.e. $\sin(2\theta) = 0.8$. This can be accomplished with $\theta = (1/2)\arcsin(0.8)$, which works out to about 0.46 radians — about 27 degrees.

A glance at the graph of the sine function reminds us that $\sin 2\theta = 0.8$ for other values of θ as well: from the first solution above we can obtain another as $\pi/2 - \theta$, about 63 degrees. So there are actually two possible ways to orient the cannon to achieve impact 400m away.

5. Compute

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2}$$

Be sure to explain why your answer is correct.

In polar coordinates ($x = r\cos\theta,\ y = r\sin\theta$) we see that (x, y) approaches zero iff $r \to 0^+$. The value of the function is $r^3(\cos^3\theta + \sin^3\theta)/r^2 = r(\cos^3\theta + \sin^3\theta)$, so as $r \to 0$, the value of this function also tends to zero (since no matter what the values of θ are, we will have $|\cos^3\theta + \sin^3\theta| \leq 1$).

6. For every function f of one variable, we can define a function F of two variables by $F(x, y) = f(x^2 + y^2)$. Compute $\dfrac{\partial^2 F}{\partial x^2}$ (in terms of f, f', and f'').

First compute $F_x(x,y) = f'(x^2+y^2)\cdot\frac{d}{dx}(x^2+y^2) = 2x\,f'(x^2+y^2)$. Then differentiate again, using the product rule: $F_{xx}(x,y) = 2f'(x,y) + 2x\,(f''(x^2+y^2)\cdot 2x)$

7. Compute the linearization of the function $F(x,y) = \ln\left(\frac{y}{x}\right)$ near $(x,y) = (3,6)$.

$L(x,y) = L(3,6) + L_x(3,6)(x-3) + L_y(3,6)(y-6)$. Now, $L_x = -1/x$ and $L_y = 1/y$ so this becomes $L(x,y) = \ln(1/2) + (-1/3)(x-3) + (1/6)(y-6)$. You can write this in other ways, e.g. as $L(x,y) = -\ln(2) - x/3 + y/6$.

Note that $F(x,y) = \ln(y) - \ln(x)$ so in this particular case we can simply linearize the two logarithms — functions of 1 variable each. In first-semester calculus you learned $\ln(x) \approx \ln(3) + (1/3)(x-3)$ for x near 3, and similarly for $\ln(y)$, so just subtract the two approximations.

8. The equation $xe^y + ye^z + ze^x = 3$ implicitly defines z as a function of x and y. Compute $\dfrac{\partial z}{\partial x}$. (Hint: don't bother trying to solve for z explicitly; this function cannot be expressed in terms of the functions you know.)

Let F be the function $F(x,y,z) = xe^y+ye^z+ze^x$; then we use the equation $F(x,y,z) = 3$ to let us think of z as a function of x and y, say $z = f(x,y)$, so that $F(x,y,f(x,y))$ is a function of x and y too but turns out to be equal to 3 for all x and y. In particular, its partial derivatives are zero:

$$0 = \frac{\partial}{\partial x}F(x,y,f(x,y)) = F_x\frac{\partial x}{\partial x} + F_y\frac{\partial y}{\partial x} + F_z\frac{\partial z}{\partial x} = F_x + F_z\frac{\partial z}{\partial x}$$

Thus $\frac{\partial z}{\partial x} = -F_x/F_z$ which we compute to be

$$-\frac{e^y + ze^x}{ye^z + e^x}.$$

9. How rapidly does the function $f(x,y,z) = \sqrt{6x-3y+2z}$ increase as we move in the direction of the vector $\langle 3,4,12\rangle$?

The direction we're talking about is that of the unit vector $u = \langle 3,4,12\rangle/||\langle 3,4,12\rangle||$ which works out to $\langle 3,4,12\rangle/13$. Then the rate of increase is the directional derivative $D_u f = \nabla f\cdot u$. In our case, $\nabla f = \frac{1}{2\sqrt{6x-3y+2x}}\langle 6,-3,2\rangle$ and we can compute a dot product $\langle 6,-3,2\rangle\cdot\langle 3,4,12\rangle = 30$, so the directional derivative is $\frac{1}{2f}(30/13) = \frac{15/13}{f(x,y,z)}$.

10. Locate and classify the local extrema of the function $f(x,y) = 4 + x^3 + y^3 - 3xy$

No restricton on the domain is given, so there are no boundary points to consider. This function is a polynomial, so there will be no points where the derivative fails to exist. Thus the only candidate points for extrema are the points where the derivative of f vanishes, i.e.

$$(0,0) = f'(x,y) = (f_x(x,y), f_y(x,y)) = (3x^2-3y, 3y^2-3x)$$

So at any critical point we must have $y = x^2$ and $x = y^2$, and thus $x = x^4$. The only such real numbers are $x = 0$ and $x = 1$, meaning the only critical points are $(0, 0)$ and $(1, 1)$.

For their classification, we must compute the second derivatives: $(f_{xx}, f_{xy}, f_{yy}) = (6x, -3, 6y)$. Thus $f_{xx}f_{yy} - (f_{xy})^2 = 36xy - 9$, which is negative at $(0, 0)$ and positive at $(1, 1)$. Since $f_{xx} > 0$ at the latter point, we conclude by the second derivative test that there is a saddle point at $(0, 0)$ and a local minimum at $(1, 1)$.

Remark: Question 1 is lightly edited from a question on the Quest review; Questions 2, 3, 4, 5, 10 are taken almost directly from the exercises in Stewart.