POSSIBLE ANSWERS

Math 408D (Rusin) Exam 1, Feb 15, 2011. Your EID:______ Name:______________________

1. Compute $\lim_{x\to\infty} \frac{x}{5} \ln\left(\frac{x-3}{x}\right)$. Since $\ln(1)=0$ this is an "$\infty \cdot 0$" type:

$$= \lim_{x\to\infty} \frac{1}{5} \cdot \frac{\ln(1-\frac{3}{x})}{1/x \to 0}$$

$$= \lim_{x\to\infty} \frac{1}{5} \cdot \frac{1/(1-\frac{3}{x}) \cdot (\frac{3}{x^2})}{(-1/x^2)} = \frac{1}{5} \cdot 1 \frac{(3)}{(-1)} = \boxed{-\frac{3}{5}}$$

2. Compute $\lim_{x\to 0^+} \left(\frac{1}{\sin(x)} - \frac{1}{1-\cos(x)}\right)$ "$\infty - \infty$ type"

$$= \lim_{x\to 0^+} \left(\frac{\sin(x)}{(\sin(x))^2} - \frac{1+\cos(x)}{1-\cos^2(x)}\right) = \lim_{x\downarrow 0} \frac{\sin x - 1 - \cos x}{(\sin x)^2}$$

As $x \downarrow 0$, $\sin x - 1 - \cos x \to -2$, and $\sin x \to 0$, so fraction $\to \boxed{-\infty}$.

3. Does the integral $\int_2^\infty \frac{x}{(3-x^2)^{1/5}}\, dx$ have a value? If so, what is it; if not, why not?

WARNING: the function is not continuous at $x = \pm\sqrt{3}$; but these points are not in the domain, so all you need is

$$\int = \lim_{T\to\infty} \int_2^T \frac{x}{(3-x^2)^{1/5}}\,dx \text{ ; with } u = 3-x^2,$$

this is $\lim_{T\to\infty} \int_{x=2}^{T} \frac{-\frac{1}{2}\,du}{u^{1/5}} = \lim_{T\to\infty} -\frac{1}{2}\frac{u^{4/5}}{(4/5)}\Big|_{x=2}^{T}$

$$= \lim_{T\to\infty} \left(-\frac{5}{8}\right)\left((3-T^2)^{4/5} - 1^{4/5}\right) = -\infty$$

4. Does the integral $\int_{-1}^{2} \frac{4}{x}\,dx$ have a value? Why or why not?

Integrand is not continuous at $x=0$, so integral is defined to be $\int_{-1}^{0} \frac{4}{x}\,dx + \int_{0}^{2} \frac{4}{x}\,dx$ (unless either of these diverges). But

$$\int_0^2 \frac{4}{x}\,dx = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^2 \frac{4}{x}\,dx = \lim_{\varepsilon \downarrow 0} \left(4\ln(2) - 4\ln(\varepsilon)\right)$$

diverges, so our integral does not exist

5. What is the limit of the sequence whose nth term is $\dfrac{6n^2}{3n+2} - \dfrac{4n^2+5}{2n-1}$?

$$a_n = \frac{(6n^2)(2n-1) - (4n^2+5)(3n+2)}{(3n+2)(2n-1)}$$

numerator $= (12n^3 - 6n^2) - (12n^3 + 8n^2 + 15n + 10)$

$= -14n^2 - 15n + 10$

denominator $= 6n^2 + n - 2$

So $a_n = \dfrac{-14 - 15/n + 10/n^2}{6 + 1/n - 2/n^2} \to -\dfrac{14}{6} = -\dfrac{7}{3}$

6. What is the limit of the sequence whose nth term is $\dfrac{n^{3n}}{(n-8)^{3n}}$?

$\log a_n = 3n \cdot \log\left(\dfrac{n}{n-8}\right)$

$= 3 \cdot \dfrac{\log\left(1 + \frac{8}{n-8}\right)}{1/n}$, so by L'Hôpital's Rule

$\lim \log(a_n) = 3 \cdot \lim \dfrac{-8/(n-8)^2}{-1/n^2} = 24$

So $\lim a_n = e^{24}$

7. I have a series $\sum_{n\geq 1} c_n$ for which the nth partial sum is $S_n = \dfrac{n^2}{3n^2+2}$. What, then, is the sum of this infinite series $\sum_{n\geq 1} c_n$?

$\sum_{n\geq 1} c_n$ is <u>defined</u> as $\lim S_n$, which is $\frac{1}{3}$!

8. Compute $\sum_{n\geq 1}\left(\dfrac{2^n+6}{4^n}\right) = \sum_{n\geq 1}\left[\left(\frac{2}{4}\right)^n + 6\cdot\left(\frac{1}{4}\right)^n\right]$, a sum of two geometric series; The sum is Thus

$$\frac{\left(\frac{2}{4}\right)}{1-\left(\frac{2}{4}\right)} + \frac{\left(\frac{6}{4}\right)}{1-\frac{1}{4}} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} + \frac{\left(\frac{3}{2}\right)}{\left(\frac{3}{4}\right)} = 1+2 = 3$$

9. If you want to estimate the value of $S = \sum_{n\geq 1}(-1)^{n-1}\dfrac{n}{2^n}$ as a decimal, you could add together the first few ($= n$) terms and then use that as your estimate for S, but of course that partial sum P ($= S_n$) will differ from S slightly. How many terms would you add together to be certain that S differs from your P by no more than 10^{-3}?

I will also give Extra Credit if you can give the correct exact value of S.

This is a strictly alternating series, so $|S - S_n| \leq |a_{n+1}| = \dfrac{n+1}{2^{n+1}}$. For $n = 8, 9, 10 \ldots$ these are $\dfrac{9}{512}, \dfrac{10}{1024}, \dfrac{11}{2048}, \dfrac{12}{4096}, \dfrac{13}{8192}, \dfrac{14}{16384}, \ldots$ and as you can see, for $n = 13$, $a_{n+1} < .001$. So 13 terms is enough.

only conditional convergence, as
$\sum \frac{1}{5\ln(n+5)} > \sum \frac{1}{5(n+5)} = \infty$ (harmonic series)

12pt Tell whether these series converge or not; if they converge, say whether they converge conditionally or absolutely. Be sure to explain how you know that what you claim is true.

10a. $\displaystyle\sum_{n\geq 1} \frac{(-1)^n}{5\ln(n+5)}$

Converges by A.S.T.:
* strictly alternates
* $\lim a_n = 0$
* terms decrease in magnitude (since $5\cdot\ln(x+5)$ is an increasing function)

10b. $\displaystyle\sum_{n\geq 1} \left(\frac{2\arctan(n)}{6}\right)^n$ Converges by: ~~[illegible]~~ Root Test:

$R = \lim \sqrt[n]{a_n} = \lim\left(\frac{\arctan(n)}{3}\right)$

$= \frac{1}{3}\cdot\left(\frac{\pi}{2}\right) < 1$

8pt 11. Find the interval of convergence for the power series $\displaystyle\sum_{n\geq 0} \left(\frac{3n+8}{5n}\right)^n x^n$. Use Ratio Test:

Series converges if $R<1$, diverges if $R>1$, where

$R = \lim\left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{\left(\frac{3n+11}{5n+5}\right)^{n+1}|x|^{n+1}}{\left(\frac{3n+8}{5n}\right)^n |x|^n} = \frac{3}{5}|x|\cdot\lim\frac{\left(1+\frac{8}{3n+3}\right)^{n+1}}{\left(1+\frac{8}{3n}\right)^n}$

$= \frac{3}{5}|x|$, so series converges if $|x| < \frac{5}{3}$

diverges if $|x| > \frac{5}{3}$

12. Find a power-series representation of $f(x) = \dfrac{x}{1-x^2}$

Recall $\displaystyle\sum_{n\geq 0} r^n = \frac{1}{1-r}$ if $|r|<1$;

Thus (taking $r = +x^2$) $\displaystyle\sum_{n\geq 0} x^{2n} = \frac{1}{1-x^2}$, so

$\displaystyle\frac{x}{1-x^2} = \sum_{n\geq 0} x^{2n+1} = x + x^3 + x^5 + \ldots$