Some examples of Trig Integrals

The first example is this: how to compute the "trigonometric integral"

$$\int \sec(x)\, dx$$

using the "recipe" I proposed to you. You can find an anti-derivative in books or online but ... how'd they do that?

So let's use the steps I proposed for you. First reduce to sines and cosines: $\sec(x) = (\sin(x))^0(\cos(x))^{-1}$, and precisely one exponent is odd, so we are advised to use the substitution $u = \sin(x)$. Then $du = \cos(x)\, dx$, so $dx = (1/\cos(x))\, du$. (The remaining corner of the little "diamond" would say $x = \arcsin(u)$ but we don't need that today.) Substituting in this expression for $dx$, our integral uses only even powers of the cosine, which is what's supposed to happen; that allows us to use the Pythagorean identity, right on schedule:

$$\int \sec(x)\, dx = \int (\cos^2(x))^{-1}\, du = \int (1 - \sin^2(x))^{-1}\, du = \int (1 - u^2)^{-1}\, du$$

As advertised, our integral is now a rational function of $u$.

So now we know to turn to Partial Fractions. No long division is necessary and the denominator factors easily as $(1 - u)(1 + u)$, so we expect the integrand to have the form

$$(1 - u^2)^{-1} = \frac{A}{1 - u} + \frac{B}{1 + u}$$

Clear denominators and you can determine the coefficients to be $A = B = 1/2$. So we may continue our transformations of the original integral:

$$\int \sec(x)\, dx = \int \frac{1}{1 - u^2}\, du = \frac{1}{2} \int \left( \frac{1}{1 - u} + \frac{1}{1 + u} \right) du = \frac{1}{2} \left( -\ln(|1 - u|) + \ln(|1 + u|) \right)$$

Plus $C$, of course.

Having finished the Calculus, we can still use some algebra: we might first rewrite this as

$$\frac{1}{2} \left( \ln \left( \left| \frac{1 + u}{1 - u} \right| \right) \right)$$

and then substitute back to get the original variable back:

$$\frac{1}{2} \left( \ln \left( \left| \frac{1 + \sin(x)}{1 - \sin(x)} \right| \right) \right)$$

There's nothing wrong with this antiderivative, but by tradition people rewrite the inner fraction by multiplying top and bottom by $1+\sin(x)$; that makes the input to the logarithm function become the square of $|(1+\sin(x))/\cos(x)| = |\sec(x)+\tan(x)|$. So using a property of logarithms, this gives the answer in the usual form,

$$\int \sec(x)\, dx = \ln\left(|\sec(x) + \tan(x)|\right) + C$$

which you should check by differentiating the right-hand side.

My second example is

$$\int 2\sin(x)\cos(x)\,dx$$

because you can use *any of the three* substitutions here — $m$ odd suggests using $u = \cos(x)$, $n$ odd suggests using $u = \sin(x)$, and $m + n$ even suggests using $u = \tan(x)$. In the last case my recipe for you would rewrite $dx = (1 + u^2)^{-1}du$, $\cos(x) = 1/\sec(x)$, and $\sin(x) = u/\sec(x)$, which would write the integral as

$$\begin{aligned}\int 2\sin(x)\cos(x)\,dx &= \int 2u(1+u^2)^{-1}/\sec^2(x)\,du = \int 2u(1+u^2)^{-1}/(1+\tan^2(x))\,du \\ &= \int 2u(1+u^2)^{-2}\,du\end{aligned}$$

which, as promised, is a rational function of $u$.

So now you might think to use Partial Fractions, but in fact this rational function is *already* in the desired form! (You might be expecting additional terms, but all those other "undetermined coefficients" will turn out to be zero.) All you need do here is a simple substitution: let $v = 1 + u^2$ so that $dv = 2u\,du$ and our integral is $\int v^{-2}\,dv = -1/v = -1/(1+u^2) = -1/(1+\tan^2(x)) = -1/\sec^2(x) = -\cos^2(x)$.

This is exactly the answer you would have gotten if you had started with the substitution $u = \cos(x)$. If you had started with the substitution $u = \sin(x)$, you would likely have ended with the answer that the antiderivative is $+\sin^2(x)$, which certainly looks different — after all, it's positive for every $x$, and the previous answer is negative for every $x$ ! Can both answers be right? Hmmm.

And what about using the identity $\sin(2x) = 2\sin(x)\cos(x)$? Then the original integral suggests using the substitution $u = 2x$ to get an antiderivative equal to $-\frac{1}{2}\cos(2x)$. How can these all be the right answer? ...