The antidifferentiation problem

$$\int \frac{6\,dx}{x\sqrt{x^2+4}}$$

gives us an opportunity to review many useful techniques.

First, since you see the square root of a quadratic polynomial, you should consider the use of a trig substitution. Please review section 7.3 until you feel comfortable that $x = 2\tan(u)$ is the most reasonable choice. After that, you should be able to confidently complete the substitution process that removes every x and dx in favor of u and du; in this example that should result in

$$3\int \frac{\sec(u)}{\tan(u)}\,du.$$

Make sure you practice substitutions (in both forms "Let $u = \ldots$" and "Let $x = \ldots$") until that step flows naturally, quickly, and accurately.

Now you have a trigonometric integral: a product (or quotient) of powers of the six basic trig functions. These are discussed in section 7.2 and you should practice a lot of these too because this is a family of functions that you will *always* be able to antidifferentiate. The first step I recommend is to write the integrand as a product of powers of sines and cosines only. (In this course we will never give a trig integration problem that cannot be so transformed, possibly using trig identities and the other tools of chapter 7.) In this example the integral may be written

$$3\int \frac{1}{\sin(u)}\,du = 3\int (\sin(u))^m(\cos(u))^n\,du$$

where in our case $m = -1$ and $n = 0$.

You have a couple of options at this point. Some students prefer to memorize the formula

$$\int \csc(u)\,du = \ln(|\csc(u) - \cot(u)|) + C$$

which is correct and fast. The only drawback to this one is that it is not at all obvious that it's even true, and it's easy to memorize wrong (especially with the negative sign in there). It might be a little easier to memorize the similar integral

$$\int \sec(u)\,du = \ln(|\sec(u) + \tan(u)|) + C$$

correctly, and our integral can be transformed to this one with the substitution $u = \frac{\pi}{2} - v$.

What I showed you in class today was an alternative that covers more such cases in a uniform way, but you should expect that any time you see a tool that can do a lot of things, it's not going to do them all equally efficiently. This approach will handle *all* these integrands $(\sin(u))^m(\cos(u))^n$ uniformly, using a substitution. Here are the three cases I showed you today:

If m is odd, let $v = \cos(u)$.
If n is odd, let $v = \sin(u)$.
If $m + n$ is even, let $v = \tan(u)$.

(Every combination of m and n is covered by one of these cases; when m and n are both odd, then any of the three substitutions may be used.) In our case we let $v = \cos(u)$ to obtain the integral

$$-3\int\frac{dv}{\sin^2(u)} = -3\int\frac{dv}{1-\cos^2(u)} = -3\int\frac{dv}{1-v^2}$$

This result is typical: you will *always* be led to a rational function of v to integrate, if you use the substitutions above (and use some trig identities).

In section 7.4 you will learn how to antidifferentiate *any* rational function, so this is cause for optimism. Right at this moment you haven't learned those tools, so I had to give you the additional piece of information that

$$\frac{1}{v^2-1} = \frac{\frac{1}{2}}{v-1} - \frac{\frac{1}{2}}{v+1}$$

which you might not have expected but which you should at least be able to verify is true. With that piece of information you should then be able to antidifferentiate

$$-3\int\frac{dv}{1-v^2} = \frac{3}{2}\int\frac{dv}{v-1} - \frac{3}{2}\int\frac{dv}{v+1} = \frac{3}{2}\ln|v-1| - \frac{3}{2}\ln|v+1| + C$$

At that point you have nearly finished the calculus part of this problem. You are obligated, after every use of Substitution, to express your answer in terms of the original variables: recall that $v = \cos(u)$ and $u = \arctan(x/2)$, so our answer is legitimately expressed as

$$\int\frac{6\,dx}{x\sqrt{x^2+4}} = \frac{3}{2}\ln|\cos(\arctan(x/2)) - 1| - \frac{3}{2}\ln|\cos(\arctan(x/2)) + 1| + C$$

Everything else we did is just an attempt to make the answer more attractive (which is to say more useful). You should certainly be comfortable drawing little right triangles to simplify any *trig(arctrig(z))* combinations; in this case $\cos(\arctan(x/2)) = 2/\sqrt{x^2+4}$. And you should be able to combine logarithms to get

$$\frac{3}{2}\ln\left|\frac{(2/\sqrt{x^2+4})-1}{(2/\sqrt{x^2+4})+1}\right|$$

Less obvious is the trickery that comes from experience: we may rewrite the argument of the logarithm in a couple of ways:

$$(A-1)/(A+1) = (A^2-1)/(A+1)^2 = (A-1)^2/(A^2-1)$$

which is a reasonable thing to do only because in our case, $|A^2 - 1| = x^2/(x^2+4)$, allowing us to use the formula $\ln(B^2) = 2\ln(|B|)$ to express our answer as either of these:

$$3\ln\left|\frac{x}{(2+\sqrt{x^2+4})}\right| = 3\ln\left|\frac{(2-\sqrt{x^2+4})}{x}\right|$$

I hope you can agree that these expressions really are algebraically equivalent, and I think you will agree that they are prettier than the answer we had at the end of the last paragraph. So you *could* get to this point, and you *should want* to get to this point, but it's not really a component of our course that you be *expected* to get this point.

I do want to emphasize another point I stressed in class today: whichever complicated technique you have used to compute your antiderivative, you really should check that it's right by using an alternative computation, namely, compute the derivative of your answer and make sure you get $6/(x\sqrt{x^2+4})$. If your algebra is not up to the task, well, maybe you should practice your algebra! But just in the privacy of your own home, you can always get convincing confirmation that you have computed a function that's equal to some other function (even if you don't see how to prove it algebraically) by simply asking your calculator to graph the difference between the two functions. A straight line means the difference in constant (including the case when the difference is zero).