Math 408S (Rusin) — EXAM 1 ANSWERS — Sep 29 2014.

1. Find the derivative of F when

$$F(x) = \int_x^{2x} (t^2 + t)\, dt.$$

Use the Fundamental Theorem of Calculus together with the Chain Rule: the derivative is $2f(2x) - f(x)$ where $f(x) = x^2 + x$. Answer is $F'(x) = 7x^2 + 3x$: **A**.

2. Calculate the antiderivative

$$I = \int (1 - \sqrt{x})(2 + \sqrt{x})\, dx.$$

Expand and integrate term-by-term: $\int (2 - x^{1/2} - x)\, dx = 2x - \frac{2}{3}x\sqrt{x} - \frac{1}{2}x^2 + C$: **A**.

3. Evaluate the integral

$$I = \int_1^9 \frac{2}{\sqrt{x}(\sqrt{x}+2)^2}\, dx.$$

Let $u = \sqrt{x}+2$ so $du = \frac{1}{2\sqrt{x}}dx$ and $I = \int_{u=3}^{u=5} (4/u^2)\, du = (-4/u)|_3^5 = (4/3) - (4/5) = 8/15$: **E**

4. Find the area of the region enclosed by the lines $x = 0$, $x = 1$, and the graphs of

$$f(x) = 4 + x - x^2 \qquad \text{and} \qquad g(x) = x + 2.$$

You MUST draw a picture to make sure the curves don't cross in this interval, but in this case the only crossings are outside the interval (find them!); indeed $f(x) > g(x)$ on this interval, so the area is $\int_0^1 (f(x) - g(x))\, dx = \int_0^1 (2 - x^2)\, dx = (2x - x^3/3)|_0^1 = 5/3$: **B**

5. Compute the antiderivative

$$I = \int 5(\sin(x))^2 (\cos(x))^3\, dx$$

Since (only) the cosine bears an odd exponent, we let $u = \sin(x)$. Then $du = \cos(x)\, dx$ and $I = \int 5\, u^2\, (1 - u^2)\, du = 5 \int (u^2 - u^4)\, du = 5(u^3/3 - u^5/5) = \frac{5}{3}u^3 - u^5 = I = \frac{5}{3}(\sin(x))^3 - (\sin(x))^5 + C$: **A**.

6. Evaluate

$$\int_1^2 \frac{(\ln(x))^2}{x^3}\,dx$$

Our integral is

$$\int_1^2 (\ln(x))^2 x^{-3}\,dx.$$

We integrate by parts: let $u = (\ln(x))^2$, and thus $dv = x^{-3}\,dx$. Then $du = 2(\ln(x)) \cdot \frac{1}{x}\,dx$ and $v = -\frac{x^{-2}}{-2}$, so our integral is

$$uv - \int v\,du = \frac{(\ln(x))^2}{-2x^2} + \frac{2}{2}\int x^{-3}\ln(x)\,dx$$

This latter integral we may also compute by parts, using $u = \ln(x)$ and $du = \frac{1}{x}\,dx$, and the same v and dv as before; so our original integral is now

$$\frac{(\ln(x))^2}{-2x^2} + \frac{\ln(x)}{-2x^2} + \frac{1}{2}\int x^{-3}\,dx = -\frac{(\ln(x))^2}{2x^2} - \frac{\ln(x)}{2x^2} - \frac{1}{4x^2}$$

Evaluating at $x = 2$ and $x = 1$ and subtracting gives the value

$$-\frac{(\ln(2))^2}{8} - \frac{\ln(2)}{8} + \frac{3}{16}$$

Let me make some remarks.

First, observe that here and elsewhere, every single $\int$ sign is balanced by a differential (dx, du, etc.) This is mandatory! The differential serves as a delimiter so you know when the description of the function has ended; it tells you what the variable of integration is; and it prevents you from miscalculating when using the Chain Rule ("u-substitution") or Product Rule ("Integration by Parts"). Likewise the differentials must "balance" in equations like $du = \frac{1}{x}\,dx$. Do it. Always. I will start docking you points on the next test if you don't do it right.

Second: $(\ln(x))^2$ is not the same as $2\ln(x)$. I know what you're thinking but you've got it wrong. The only numbers A for which A^2 is the same as $2A$ are $A = 0$ and $A = 2$!

Third: Integration by Parts is a useful algorithm. It's very mechanical, so learn how to carry it out! After you decide on what u shall be, it's all on auto-pilot, if you've practiced. (Occasionally it's hard to get v from dv but in that case you've probably chosen a bad u; try again.) Yes, some creativity can be required to select a good u, but I will not give you such tricky problems on a test. Keep practicing until you get this.

Fourth: since you are spending so much time computing an antiderivative, it's well worth your time to differentiate your putative answer and see that you get the original integrand back.

Finally: remember that you're not just scribbling your own thoughts here: you're presenting an answer for me to read. I'm going to start at the top left and read across and down, basically reading aloud every symbol and nothing else. What I say in the process had better sound like English, or it's gibberish. So please proceed linearly down the page. Write clearly. Put "=" signs between things that are equal (and nowhere else). A few English words ("Let", "Thus", "But", etc.) go a long way to making something readable. Some of you are already doing this. The rest of you should, too.

7. Compute

$$\int_1^5 \sqrt{-x^2+6x-5}\,dx.$$

Hint: Complete the square.

This is the area under the curve $y = \sqrt{-x^2+6x-5}$, i.e. the top half of the curve $(x-3)^2+y^2=4$. That's a circle of radius 2, so the enclosed area is 2π. Done! (Does that argument sound familiar?...)

None of you solved the problem this way, and that's fine. Here's what I expected you to do. "Completing the square" means rewriting x^2-6x+5 as $(x-3)^2-4$, so that we are computing

$$\int_1^5 \sqrt{4-(x-3)^2}\,dx.$$

(If you got stuck already, try starting with a substitution like $u = x-3$.) The form of this integral suggests a trigonometric substitution $x-3 = 2\sin(\theta)$, so that $dx = 2\cos(\theta)\,d\theta$ and the integrand is $\sqrt{4-4\sin^2\theta} = |2\cos(\theta)|$. The interval $[1,5]$ for x corresponds to the interval $[-\pi/2, \pi/2]$ of θ, so our integral has become

$$\int_{-\pi/2}^{\pi/2} 2|\cos(\theta)|\cdot 2\cos(\theta)\,d\theta$$

We can drop the absolute value signs since $\cos(\theta) > 0$ on this interval. Then since $2\cos^2\theta = 1+\cos(2\theta)$ we compute our integral as

$$2\int_{-pi/2}^{\pi/2}(1+\cos(2\theta))\,d\theta = 2(\theta+\frac{1}{2}\sin(2\theta))|_{-\pi/2}^{\pi/2} = 2\pi \quad .$$

As in the previous problem I really want to encourage you to be more organized and clear in your writing; it will help YOU not get lost! And there are not that many trig substitutions we ever use; learn which is which (and learn your trig identities!)

8. Sketch the region bounded by the curves $y = \ln(x), x = 1, x = e$, and the x axis. Then compute the volume of the region obtained by rotating this figure around the y-axis.

I'm very sorry to have given you papers with typos on them! As I promised, I accepted many different answers in case you had already started the problem with other bounds for the region. This is a great example of why it's to your advantage to sketch the region you're rotating – complete with labeled lines and curves! (Many, many of you need to learn what the graph of the logarithm function looks like ...) It's also to your advantage to draw one slice/shell, clearly indicating some line segments whose lengths give you the relevant data for computing volumes (radii, thicknesses, heights, etc.)

To compute the volume using slices, slice across the y-axis to see a bunch of washers, the one at a height of y having radii equal to e on the outside and $x = e^y$ on the inside. So the volume is

$$\int_{y=0}^{y=1} \pi(e^2 - (e^y)^2)\, dy = \pi \int_0^1 (e^2 - e^{2y})\, dy$$

The antiderivative is $e^2 y - \frac{e^{2y}}{2}$; evaluating at 0 and 1 gives a volume of $\pi(e^2 + 1)/2$.

To compute the volume using shells around the y-axis, note that the one that's a distance of x away from the axis has radius x (duh) and height $y - 0 = \ln(x)$, so the volume is

$$\int_{x=1}^{x=e} 2\pi \cdot x \cdot \ln(x)\, dx = 2\pi \cdot \left(\frac{x^2}{2}\ln(x) - \frac{x^2}{4}\right)|_1^e$$

Substitute and subtract and get the same volume as above (of course!)

If you'd like me to help you see how to compute the volumes of other spun regions related to this problem just ask.