Here is a procedure that will allow you to compute any trig integral (by which I mean an integral in which the integrand is a product of powers of $\sin(x), \cos(x), \tan(x)$, or any of the other three basic trig functions of x.

Please remember that although this process will work, it is NOT always the most efficient way to obtain the answer. (For example, you will notice that I only use the "double angle trick" for one specific case, even though it works quite well in other cases.) Be willing to be creative! There are plenty of techniques you can try besides what I suggest here (complex exponentials! Fourier series! etc.) and of course you could simply believe the formulas others have worked out for you, but I wanted you to see at least one way that people have been able to get those formulas in the first place.

OK, so here are the steps.

1. Use the definitions of the trig functions to express the integrand as $\sin(x)^k \cos(x)^m$ for some integers k and m (possibly negative).

2. In the special case that $k = 0$ and m is even and positive, the steps below don't actually help (I encourage you to try them and see why!) so for these cases we have to use the double-angle reduction: since $\cos(x)^2 = (1 + \cos(2x))/2$ we can compute these anti-derivatives as $\int \cos(x)^m \, dx = (1/2)^{m/2} \int (1 + \cos(2x))^{m/2}$, which we then expand and compute by the steps outlined below. It's not a bad idea simply to write down the results of some of these calculations once and for all:

$$\int \cos^2(x)\, dx = \frac{1}{2}(x + \cos(x)\sin(x))$$

$$\int \cos^4(x)\, dx = \frac{1}{4}(\frac{3}{2}x + \frac{3}{2}\cos(x)\sin(x) + \cos(x)^3\sin(x))$$

$$\int \cos^6(x)\, dx = \frac{1}{8}(\frac{5}{2}x + \frac{5}{2}\cos(x)\sin(x) + \frac{5}{3}\cos(x)^3\sin(x) + \frac{4}{3}\cos(x)^5\sin(x))$$

3. Now choose one of these three u-substitutions: if k is odd, let $u = \cos(x)$; if m is odd, let $u = \sin(x)$; if $k + m$ is even, let $u = \tan(x)$. One of these three will apply in each circumstance; if k and m are both odd, then you can use any of the three substitutions.

4. Next, perform the substitution. In the third case it is probably easier to first replace each $\sin(x)$ by $\tan(x)\cos(x)$ and then to replace each $\cos(x)$ by $(\sec(x))^{-1}$. Then in each case you will need to change the dx to the appropriate multiple of du, which will slightly transform the integral. Then the rest of the integrand can be replaced by a power of u times a power of $1 - u^2$ (in the first two cases) or $1 + u^2$ (in the third). That is, you will now have a *rational function* of u as your integrand.

Specifically, these substitutions will transform the integral

$$\int (\sin(x))^k (\cos(x))^m \, dx$$

to, respectively, one of these three:

$$-\int (1-u^2)^{(k-1)/2} u^m \, du \quad (case\ 1)$$

$$\int u^k (1-u^2)^{(m-1)/2} \, du \quad (case\ 2)$$

$$\int u^k (1+u^2)^{-(m+k+2)/2} \, du \quad (case\ 3)$$

5. Now you have to antidifferentiate a rational function. Of the three cases shown above, the only types that require expansion using Partial Fractions would be those with powers of $1 \pm u^2$ in the denominator (possibly along with powers of u), i.e. rational functions of the following forms:

$$\frac{u^a}{(1-u^2)^b} \qquad \frac{u^a}{(1+u^2)^b}$$

where a and $b > 0$ are integers. I promise to never give you any case where $|a| > 2$ or $b > 2$ (although they require no additional theory or tricks, just longer computations), so let me show you the decompositions you would need:

$$\frac{1}{(1-u^2)} = \frac{1}{2(1-u)} + \frac{1}{2(1+u)}$$

$$\frac{u}{(1-u^2)} = \frac{1}{2(1-u)} - \frac{1}{2(1+u)}$$

$$\frac{u^2}{(1-u^2)} = -1 + \frac{1}{2(1-u)} + \frac{1}{2(1+u)}$$

$$\frac{1}{u(1-u^2)} = \frac{1}{2(1-u)} - \frac{1}{2(1+u)} + \frac{1}{u}$$

$$\frac{1}{u^2(1-u^2)} = \frac{1}{2(1-u)} + \frac{1}{2(1+u)} + \frac{1}{u^2}$$

$$\frac{1}{(1-u^2)^2} = \frac{1}{4(1-u)} + \frac{1}{4(1+u)} + \frac{1}{4(1-u)^2} + \frac{1}{4(1+u)^2}$$

$$\frac{u}{(1-u^2)^2} = \frac{1}{4(1-u)^2} - \frac{1}{4(1+u)^2}$$

$$\frac{u^2}{(1-u^2)^2} = -\frac{1}{4(1-u)} - \frac{1}{4(1+u)} + \frac{1}{4(1-u)^2} + \frac{1}{4(1+u)^2}$$

$$\frac{1}{u(1-u^2)^2} = \frac{1}{2(1-u)} - \frac{1}{2(1+u)} + \frac{1}{4(1-u)^2} - \frac{1}{4(1+u)^2} + \frac{1}{u}$$

$$\frac{1}{u^2(1-u^2)^2} = \frac{3}{4(1-u)} + \frac{3}{4(1+u)} + \frac{1}{4(1-u)^2} + \frac{1}{4(1+u)^2} + \frac{1}{u^2}$$

$$\frac{u^2}{u^2+1} = 1 - \frac{1}{u^2+1}$$

$$\frac{1}{u(u^2+1)} = \frac{1}{u} - \frac{u}{u^2+1}$$

$$\frac{1}{u^2(u^2+1)} = \frac{1}{u^2} - \frac{1}{u^2+1}$$

$$\frac{u^2}{(u^2+1)^2} = \frac{1}{u^2+1} - \frac{1}{(u^2+1)^2}$$

$$\frac{1}{u(u^2+1)^2} = \frac{1}{u} - \frac{u}{u^2+1} - \frac{u}{(u^2+1)^2}$$

$$\frac{1}{u^2(u^2+1)^2} = \frac{1}{u^2} - \frac{1}{u^2+1} - \frac{1}{(u^2+1)^2}$$

6. So ultimately we need only to be able to compute the following integrals:

$$\int \frac{1}{u}\,du = \ln(|u|) \qquad \int \frac{1}{u^2}\,du = -\frac{1}{u} \qquad \int \frac{1}{1+u}\,du = \ln(|1+u|) \qquad \int \frac{1}{1-u}\,du = -\ln(|1+u|)$$

$$\int \frac{1}{(1+u)^2}\,du = -\frac{1}{1+u} \qquad \int \frac{1}{(1-u)^2}\,du = \frac{1}{1-u}$$

$$\int \frac{1}{u^2+1}\,du = \arctan(u) \qquad \int \frac{u}{u^2+1}\,du = \ln(u^2+1)$$

$$\int \frac{1}{(u^2+1)^2}\,du = \frac{u}{2(u^2+1)} + \frac{1}{2}\arctan(u) \qquad \int \frac{u}{(u^2+1)^2}\,du = -\frac{1}{2(u^2+1)}$$

which you ought to be able to compute yourself, using substitutions like $v = 1-u$ or $v = 1+u^2$ or $u = \tan(z)$. (With larger b and $|a|$ you might also have to integrate higher powers of $1/(1+u^2)$, using the same substitution $u = \tan(z)$, which would in turn lead to the problem of integrating even powers of $\cos(z)$, which we treated in step 2.)