BENNETT DIFFERENTIAL EQUATIONS EXAM 2019 — ANSWERS

1. If $f(x)$ is the function defined by

$$f'(x) = \frac{f(x)}{4f(x)+3x-3} \qquad \text{and} \qquad f(0) = 1 \ ,$$

what is the value of $f(3)$? (Partial credit will be given for a numerical estimate of this value, with more credit for a closer approximation.)

ANSWER: To give a numerical estimate of $f(3)$, you are welcome to use Euler's method (which would be laborious without a calculator!) or a graph of direction fields, or a series solution (which actually does not converge at $x = 3!$), etc. But we can solve this differential equation explicitly.

Write y for $f(x)$ and invert the derivative to view the problem as the first-order linear equation

$$\frac{dx}{dy} = \frac{4y+3x-3}{y} = \left(\frac{3}{y}\right)x + \left(4 - \frac{3}{y}\right)$$

An integrating factor is y^{-3} from which we learn the general solution is $x = 1-2y+Cy^3$. The initial condition forces $C = 1$. Then the value of y corresponding to $x = 3$ is a root of $y^3 - 2y - 2 = 0$. You can estimate the root as being between y=1.5 and y=2 from a graph of this cubic, or you may want to use Newton's Method to improve your estimate, e.g. from $y_0 = 2$ we get $y_1 = 2 - \frac{2}{10} = 1.8$. The actual root is $f(3) = 1.769292354\ldots$ (It's $a/3 + 2/a$ where $a = (27 + 3\sqrt{57})^{1/3}$!)

2. For some functions $A(x)$ and $B(x)$, the set of solutions of the differential equation

$$y' = A(x)y + B(x)$$

includes both the tangent function $y = \tan(x)$ and the cosine function $y = \cos(x)$. What is the solution to the initial-value problem

$$y' = A(x)y + B(x), \qquad y(0) = \pi \ ?$$

ANSWER: The function $\cos(x) - \tan(x)$ is a solution of the homogeneous problem $y' = Ay$, so the general solution to the inhomogeneous equation is $y = \tan(x) + C\,(\cos(x) - \tan(x))$. Note that when $x = 0$, $y = C$. So for the given problem we need $C = \pi$ and thus $y = \tan(x) + \pi\,(\cos(x) - \tan(x))$.

One could also determine $A(x)$ and $B(x)$ from 2 equations in these 2 unknowns, but this is neither necessary nor helpful.

3. Find a solution to the partial differential equation

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = z$$

which is not a polynomial in x and y. For extra credit give the general solution.

ANSWER: The general solution is $z = x\,f(xy)$ for any differentiable function f, so for example $z = x$ is a polynomial solution and $z = 1/y$ is a non-polynomial solution. (You could find these e.g. using Separation of Variables.)

The general solution can be found with the "method of characteristics", as follows. We attempt to transform this into a one-variable differential equation (an ODE) with a coordinate change, starting by finding a new coordinate u with $u_x/u_y = y/x$; on the curve where this u stays constant, the left side computes $-dy/dx$ by Implicit Differentiation, leaving us with the separable differential equation $-dy/y = dx/x$, whose solution is $xy = C$. So we use $u = xy$ as a new coordinate. You can for example use $v = x$ as a second coordinate.

To clarify the point of those calculations, observe the result of rewriting the differential equation in (u, v) coordinates using the Chain Rule:

$$z_x = (y \cdot z_u + 1 \cdot z_v) \qquad z_y = (x \cdot z_u + 0 \cdot z_v)$$

so we simply get $xz_v = z$, i.e. $vz_v = z$, an ODE. This equation is separable and has general solution $\ln(z) = \ln(v) + C$ where $dC/dv = 0$, that is, C is a function of u alone. Exponentiate to see $z = vf(u)$ for some function of u, as claimed.

4. Find a (nonzero) solution of the linear differential equation

$$5x^2y'' + x(1 + x)y' - y = 0$$

ANSWER: One solution is $y_0 = x^{-1/5}e^{-x/5}$. One may then obtain all other solutions using Variation of Parameters, but writing them down requires an antiderivative which is not expressible in simple terms, so effectively this solution is the only one you will be able to find.

This is not quite a Cauchy-Euler equation because of the factor $(1+x)$ and may not be solvable by series because of the factor $5x^2$. Nonetheless the "Method of Frobenius" still allows us to find a series solution.

As with the Cauchy-Euler problem we consider solutions of the form $y(x) = x^r z(x)$ for appropriate choices of r, namely the roots of the "indicial equation" $5r(r - 1) + 1r - 1 = (5r + 1)(r - 1)$.

When $r = -1/5$ we find (after simplifying) that the function $z(x)$ must satisfy the equation

$$5xz'' + (x - 1)z' - \frac{1}{5}z = 0$$

If we posit that z has a series expansion $z(x) = \sum a_n x^n$ then the differential equation forces the recurrence $a_{n+1} = -\dfrac{a_n}{5(n+1)}$, from which we recognize the solution: $a_n = a_0(\frac{-1}{5})^n/n!$. Therefore this series converges to $a_0 e^{-x/5}$, giving the solution indicated. (It is also possible to separate the terms of the defining differential equation as $x(5z' - z)' = (5z' - z)$ which is easily solved with two integrations.)

The other root $r = 1$ similarly has a series solution but the recurrence is $a_{n+1} = -a_n/(5n+11)$ which gives a series with a definite pattern but no easy summation in terms of elementary functions.

One student also noted that upon dividing the given equation by $5x^2$ we obtain an equation which may be expressed as $y'' + uy' + u'y = 0$ where $u = \frac{1}{5}(1 + \frac{1}{x})$. This, in turn, may be rewritten as $(y' + uy)' = 0$, which implies $y' + uy$ will be constant. So we have only a first-order linear ODE to solve, and its integrating factor is easy to find (it happens to be the same function y_0 mentioned above). To obtain the full solution one must evaluate a complicated antiderivative but the special case $y' + uy = 0$ has y_0 as a solution, as required.

5. Does every solution of the differential equation $y'' + e^x y = 0$ stay bounded as $x \to \infty$?

ANSWER: Yes. Simply observe that

$$(y^2 + (y')^2 e^{-x})' = 2e^{-x} y'(y'' + e^x y) - (y')^2 e^{-x} = -(y')^2 e^{-x}.$$

which is negative for every x. That means $y^2 + (y')^2 e^{-x}$ is a decreasing function, and so for every $x > 0$ we have $y(x)^2 < y(x)^2 + (y'(x))^2 e^{-x} < y(0)^2 + (y'(0))^2$.

(This was question B2 on the 1966 Putnam exam.)

Several students noted that since $e^x > 0$ for all x, the function would be concave up whenever negative and concave down whenever positive, suggesting oscillatory behaviour: one positive local maximum, then a zero, then a negative local minimum, another zero, etc. What is difficult to rule out this way is the possibility that the local maxima steadily rise without bound. With $y'' = -e^x y$, that would imply that the curvature would increase rather dramatically as we move from one "hump" to the next, but that is not obviously a contradiction; it would simply imply that the gaps between successive zeros are rapidly decreasing. (To get a contradiction would, I think, require showing that the gaps between zeros are small enough that the sum of the gaps is finite.)