BENNETT LINEAR ALGEBRA PRIZE EXAM   Dec 09 2020

Submit your solutions *with all work shown*, **by 8pm (Austin time) as an email attachment to rusin@math.utexas.edu. During the exam you must abide the rules previously sent via email.**

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1. Express the function $f(x) = \sin^3(x)\cos^2(x)$ as a linear combination of the functions $\sin(nx)$ and $\cos(nx)$, for $n = 0, 1, 2, \ldots$. (Possible hint: the decomposition should give a function which at the very least takes on the right values of f for a few well-known values of x.)

2. An *Inverted Pascal's Triangle* (IPC) is an arrangement of 55 numbers in 10 rows; the nth row (counting from bottom up) has n numbers in it, and every entry in the IPC is the sum of the two numbers diagonally above it. If only some of the 55 cells have numbers in them and the others are left blank, it may be possible to determine what numbers belong in the empty cells, according to the rules. What is the smallest number of entries that must be filled in to enable us to determine the missing entries?

3. The set M_n of all $n \times n$ matrices with real entrties forms a vector space. Find a basis of M_3 consisting of elements of M_3 that all commute with one another, or prove that no such basis exists.

4. There are 2^{16} four-by-four matrices whose entries are all 1's and 3's. Find the average of all their determinants.

5. Suppose that $L : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear transformation and that $\ker(L) = \{0\}$ (that is, the null space of L consists only of the zero vector in $\mathbf{R}^2$). First, show that there is a linear transformation $K : \mathbf{R}^3 \to \mathbf{R}^2$ such that $K(L(v)) = v$ for every $v \in \mathbf{R}^2$ (that is, $K \circ L = I_2$, the identity map on $\mathbf{R}^2$. Then decide which of the following is true:

(a) $L \circ K$ must equal I_3, the identity map on $\mathbf{R}^3$

(b) $L \circ K$ cannot equal I_3

(c) Whether $L \circ K = I_3$ or not depends on the choices of K and L.

Answers will soon appear at `http://www.math.utexas.edu/users/rusin/Bennett/`