BENNETT LINEAR ALGEBRA PRIZE EXAM Dec 07 2021

1. Let V be the set of functions of the form $f(x) = p(x)\, e^{-x}$ where $p(x)$ is a polynomial of degree at most 4. Note that the derivative $D(f) = f'$ of every element of V is also a member of V. Find a basis $\mathcal{B}$ of V, and then find the matrix representation of D with respect to this basis. Is there a basis for which the matrix representing D is a diagonal matrix?

ANSWER: A natural basis might consist of the five elements $\mathcal{B} = \{f_0, f_1, f_2, f_3, f_4\}$ where each $f_k(x) = x^k e^{-x}$. With the product rule we compute $D(f_k) = k\, f_{k-1} - f_k$ so the matrix representing D with respect to this (ordered) basis is

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

In any other basis, the matrix representation could scarcely be any simpler, and certainly in no basis is the matrix representation diagonal, because there is not a basis of eigenvectors: the only eigenvalue is -1 and the (-1)-eigenspace is one-dimensional. These facts can be deduced in two ways. On the one hand, we can see from *this* matrix, which is already upper-diagonal, that the only eigenvalue for this linear transformation is -1 but the eigenspace consists only of the span of f_0. Alternatively, we can work in V itself: an eigenvector of D would be a function f whose derivative $D(f) = f'$ is a scalar multiple of itself; the only such functions have the form $f(x) = Ae^{bx}$ for some A and b, and of these the only ones in V are the functions $f(x) = Ae^{-x}$ which are, again, the scalar multiples of f_0.

2. Suppose u and v are vectors in $\mathbf{R}^3$ and that we know these lengths: $||u|| = 3$, $||u+v|| = 4$, and $||u - v|| = 6$. What is the length $||v||$ of the vector v?

ANSWER: Recall that for any vector w we have $||w||^2 = w \cdot w$; in particular for any scalars a and b we have $||au + bv||^2 = (au + bv) \cdot (au + bv) = a^2||u||^2 + 2ab(u \cdot v) + b^2||v||^2$. Thus each piece of information about the length of a linear combination of the vectors u and v determines a linear relation among the three numbers $X = ||u||^2$, $Y = u \cdot v$, and $Z = ||v||^2$. Three such pieces of information then leave us with three equations in three unknowns, which we can (usually) expect to have a unique solution (X, Y, Z). In this problem the equations are quite trivial and show that $(X, Y, Z) = (9, -5, 17)$, so that $||v|| = \sqrt{17}$. (There is a unique solution precisely when the determinant of the coefficient matrix is nonzero; that coefficient matrix is essentially the Vandermonde matrix of the three ratios $a : b$, and hence the determinant is nonzero as long as the pairs (a, b) of coefficients presented are not collinear.)

It should not be surprising that it is possible to deduce any length from any other three lengths: this is essentially the "side-side-side" theorem from high-school geometry!

3. Suppose A is a 2×2 matrix which satisfies $A^3 = A$. Show that A^2 must be equal to one (or more) of 0, I, A, or $-A$.

ANSWER: If A is invertible, then we may multiply both sides of the equation $A^3 = A$ by A^{-1} to deduce $A^2 = I$ (which conversely implies both that A is invertible and that $A^3 = A$!)

If instead A is not invertible, then its determinant d must be zero. It follows that the characteristic polynomial of A is simply $X^2 - cX + 0$ (where c is the trace of A). Now, the Hamilton-Cayley theorem asserts that A satisfies its characteristic polynomial, i.e. that $A^2 - cA = 0$ so that $A^2 = cA$. If c is either $+1$ or -1 we are done. Otherwise, we have a string of identities

$$A = A^3 = A(A^2) = A(cA) = c(A^2) = c(cA) = (c^2)A$$

from which we deduce that $(c^2 - 1)A = 0$. But $c^2 - 1$ is nonzero, so we may multiply by its reciprocal and conclude that A itself (not merely its square!) is the zero matrix.

So in every case, A^2 is equal to one of the four possibilities given.

(The converse is also true: if $A = 0$, or if A^2 is equal either to I or A or $-A$, then $A^3 = A$.)

There are more such matrices than you might imagine. For example, the matrices for which $A^2 = I$ are precisely I and $-I$ *and* all the matrices $P^{-1}DP$ where P is any invertible matrix and D is the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; and there are doubly-infinite families of the other sets of matrices too!

The restriction to 2×2 matrices is *almost* necessary, if you are familiar with the concept of a matrix's *minimal polynomial*. It is a consequence of the Division Algorithm for Polynomials that, of all the nonzero polynomial equations which is satisfied by a matrix

M, the one of lowest degree divides all the others. In this problem, the minimal polynomial must then be a divisor of $X^3 - X = X(X+1)(X-1)$, and so it is one of the following:

$$X, \quad X+1, \quad X-1, \quad X(X+1), \quad X(X-1), \quad (X+1)(X-1), \quad X(X+1)(X-1).$$

In most of these cases we may draw a conclusion about A, namely that

$$A = 0, \quad A = -I, \quad A = +I, \quad A^2 = -A, \quad A^2 = +A, \quad A^2 = I$$

respectively. The only case not covered is the last one, in which $X^3 - X$ really is the *minimal* polynomial of A. But by Hamilton-Cayley, the minimal polynomial must divide the characteristic polynomial as well, so this would force the matrix to have at least 3 rows and columns; indeed, it would force $0, 1$, and -1 all to be eigenvalues. A relevant example would be the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

4. Recall that the *trace* $\mathrm{Tr}(M)$ of a real, $n \times n$ matrix M is the sum of the diagonal entries of M.

(a) Find such a matrix B for which $\mathrm{Tr}(B^2) < 0$

(b) Show that if C is symmetric then $\mathrm{Tr}(C^2) \geq 0$

(c) Show that if M has n distinct real eigenvalues then $\mathrm{Tr}(M^2) \geq 0$

ANSWER: There are many solutions to (a), even just for 2×2 matrices; we simply need the off-diagonal entries to be "large" and of opposite sign, e.g. the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ representing a 180-degree rotation around the origin in $\mathbf{R}^2$, whose square is obviously $-I$.

For (b), notice that the diagonal entries of C^2 are sums of squares: the (i,j) entry of the product of any two matrices M and N is $\sum_{k=1}^n M_{i,k}N_{k,j}$, so the (i,i) entry of C^2 is $\sum_{k=1}^n C_{i,k}C_{k,i}$, which is equal to $\sum_{k=1}^n (C_{i,k})^2$ since C is symmetric. That makes each diagonal entry of C^2 non-negative, and hence so is the sum of all those entries.

For (c) recall that a matrix with distinct real eigenvalues is diagonalizable, that is, there is a diagonal matrix D (whose diagonal entries are those eigenvalues) and an invertible matrix P (the change-of-basis matrix to the basis consisting of the corresponding eigenvectors) such that $M = P^{-1}DP$. But then $M^2 = P^{-1}D^2P$ so M^2 and D^2 have equal traces; but the trace of the diagonal matrix D^2 is clearly non-negative (as in part (b)). So the trace of M^2 is also non-negative.

5. Find the rank, and a basis for the null space (=kernel), of the $n \times n$ matrix M whose (i, j) entry is $M_{ij} = (i + j - 2)^2$. For example, for $n = 4$ the matrix M is

$$\begin{pmatrix} 0 & 1 & 4 & 9 \\ 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \end{pmatrix}$$

ANSWER: Of course when $n = 1$ the rank is zero and the kernel is all of $\mathbf{R}^1$. It is easy enough to check manually that when $n = 2$ or 3, the matrix M has a nonzero determinant, so the rank is n and the nullspace consists only of $\{0\}$. In particular, this means there is no linear relation among the rows of the 3×3 submatrix in the upper-left corner of the matrix M when $n \geq 4$; hence the top three rows of M are linearly independent for all these n, making their ranks at least 3.

But it turns out that for $n \geq 4$, the rank is also at most 3, because we can find $n - 3$ linearly independent elements in the kernel of M, namely, vectors of the form

$$v_i = \langle 0, 0, \ldots, 0, 1, -3, 3, -1, 0, 0, \ldots, 0 \rangle$$

for $i = 1, 2, \ldots, n - 3$, where there are $i - 1$ zeros at the beginning and $n - 3 - i$ zeros at the end.

The reason for this is that if a, b, c, d are any four consecutive perfect squares, then $a - 3b + 3c - d = 0$. This is easily checked as follows: if a is the square of some integer k then we have $a = k^2, b = (k+1)^2, c = (k+2)^2$ and $d = (k+3)^2$; expanding $d - 3c + 3b - a$ we get $(k^2 + 6k + 9) - 3(k^2 + 4k + 4) + 3(k^2 + 2k + 1) - k^2 = 0$, as desired.

A similar proof would apply to a matrix filled with consecutive cubes, consecutive fourth powers, etc. Indeed the example of a matrix filled with consecutive *first* powers was given as an example in the list of eligible topics for this competition! (The archetype of the kernel vector in that case is $\langle 1, -2, 1 \rangle$, and if you think you see a binomial pattern in these two examples, you're right!)