1. (20 pts.) Compute the following limits

(i) $\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^{3n}$

(ii) $\lim_{x\to 0} x^{-1}\int_3^{3+x}\cos(\pi y^2)\,dy$

(iii) $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{3^k}{k!}$

(iv) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k\pi}{n^2}\sin\left(\frac{k\pi}{n}\right)$

(v) $\lim_{x\to\infty} x\left(1-e^{-(1/x)}\right)$

ANSWER:

(i) Exponentiation of real numbers a^b (with $a>0$) may be written as $e^{b\ln(a)}$; since the exponential and logarithm functions are continuous we can then compute $\lim a^b$ as $e^{\lim(b\ln a)}$. In our case this requires that we compute $\lim_{n\to\infty} 3n\ln(1-(2/n))$. We may substitute $n=1/u$; then we need the limit as $u\to 0^+$ of $3\ln(1-2u)/u$. With one application of L'Hôptial's Rule this limit is seen to be -6. So the original limit evaluates to e^{-6}.

(ii) Writing this as $\lim_{x\to 0}\frac{F(x)}{x}$ we see that this again may be computed using L'Hôptial's Rule (since clearly the integral $F(x)$ will vanish when $x=0$). But $F'(x)=\cos(\pi(3+x)^2)$ by the Fundamental Theorem of Calculus, so the limit involved in L'Hôptial's Rule is simply $\cos(9\pi)=-1$.

(iii) This is the limit of the partial sums of an infinite series $\sum_{k\geq 0}3^k/k!$. But we recognize this as the Taylor series of the exponential function, evaluated at $x=3$. Hence the value of this limit is e^3.

(iv) This may be written $\lim_{n\to\infty}\sum_{k=1}^{n}F(x_k)\Delta x$, where $F(x)=\frac{1}{\pi}x\sin(x)$, $x_k=k\pi/n$, and $\Delta x=x_k-x_{k-1}$ (which is π/n). But such a sum is a Riemann sum associated to the integral $\int_0^\pi F(x)\,dx$, using the right-end end points to represent each of the n subintervals into which the interval $[0,\pi]$ has been divided. Since the limit of the Riemann sum defines the value of the integral, our limit is $\int_0^\pi F(x)\,dx = \frac{1}{\pi}\int_0^\pi x\sin(x)\,dx$. We evaluate an antiderivative using Integration By Parts, to get $-x\cos(x)+\sin(x)+C$; using the Fundamental Theorem of Calculus the value of the integral is π and so the original limit is 1.

(v) As in the first limit we substitute $u = 1/x$ to get $\lim_{u\to 0^+}(1-e^{-u})/u$ and then use L'Hôpital's Rule to see the limit equals 1.

2. (10 pts.) A perfectly spherical apple of radius 3 centimeters is centered at the origin. A worm crawls along the x-axis, eating every bit of the apple whose distance from the x-axis is less than 1 centimeter. Find the volume of the remaining uneaten portion of the apple.

ANSWER: We can calculate the volume with the "method of washers", that is, the volume is the integral $\int_{-3}^{3} A(x)\,dx$ of the cross-sectional area of portion that the worm did *not* eat of the slice of the apple at a given x coordinate. Note that $A(x) = 0$ when x is close to ± 3; in fact the worm eats the entirety of the slice unless $|x| \le \sqrt{8}$. Then, for x in this interval, the uneaten portion is an annulus (a "washer") whose inner radius is always 1cm and whose outer radius is $\sqrt{9-x^2}$. Thus the area $A(x)$ of the uneaten slice is $\pi(9-x^2) - \pi$ cm^2. It follows that the volume of the uneaten portion is

$$\pi\int_{-\sqrt{8}}^{\sqrt{8}} (8-x^2)\,dx = \frac{64\sqrt{2}\pi}{3}\text{cm}^3$$

The volume can also be computed by the method of cylindrical shells.

3. (10 pts.) Compute $\displaystyle\int_0^\infty \frac{1}{(1+x^2)^3}\,dx$.

ANSWER: This is an improper integral, so we must compute an antiderivate and study its endpoint behaviour. Using the substitution $x = \tan(\theta)$ the integral becomes $\int \cos^4(\theta)\,d\theta$, which we evaluate with the customary trigonometric idenities:

$$\begin{aligned}\cos(\theta)^4 &= \frac{1}{4}(1+\cos(2\theta))^2 \\ &= \frac{1}{4}\left(1 + 2\cos(2\theta) + \frac{1+\cos(4\theta)}{2}\right) \\ &= \frac{1}{32}\left(12\theta + 8\sin(2\theta) + \sin(4\theta)\right)\end{aligned}$$

With several applications of the double-angle formulas, this may be written

$$\frac{1}{8}\left(3\theta + 4\cos(\theta)\sin(\theta) + 2\cot^2(\theta)\sin(\theta)\right) \ .$$

Substituting back $\sin(\theta) = x/\sqrt{1+x^2}$ and $\cos(\theta) = 1/\sqrt{1+x^2}$ gives

$$\int \frac{1}{(1+x^2)^3}\, dx = \frac{1}{8}\left(3\arctan(x) + \frac{3x}{1+x^2} + \frac{2x}{(1+x^2)^3}\right)$$

Taking now the integral over any interval $[0, T]$ and letting $T \to +\infty$ gives the value of the integral as $3\pi/16$.

4. (10 pts.) Line L is the intersection of the planes $2x + 2y + z = 4$ and $x - y - z = 1$. There are two spheres of radius 3 which pass through the origin and whose centers lie on L. Find the equations of the spheres.

ANSWER: It is easier to use a parametric description of this line. The normal vectors of the two planes are $\langle 2, 2, 1\rangle$ and $\langle 1, -1, -1\rangle$ respectively; the cross product of these two vectors, namely $\langle 1, -3, 4\rangle$, is then parallel to both the planes and hence to their intersection, the line L. Pick any point on the line (say, $(1, 2, -2)$) and add multiples of this vector to it to get a parameterization:

$$L = \{(1+t,\ 2-3t,\ -2+4t) \,|\, t \in \mathbf{R}\ \}$$

So now we need only to find the values of t for which a sphere of radius 3 with such a center passes through the origin, that is, the values of t for which this point is three units away from $(0, 0, 0)$. Clearly this happens iff $(1+t)^2 + (2-3t)^2 + (-2+4t)^2 = 9$. That's a quadratic equation with roots $t = 0$ and $t = 1$. So the two good centers on L are $(1, 2, -2)$ and $(2, -1, 2)$ (which obviously are indeed a distance of 3 from the origin). Then the spheres are give by the equations

$$(x-1)^2 + (y-2)^2 + (z+2)^2 = 9 \qquad \text{and} \qquad (x-2)^2 + (y+1)^2 + (z-2)^2 = 9$$